Aoi

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Elements of Mathematical and Logical Reasoning

An Introduction

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A slight ordering of my interior begins to take place and I need nothing more, for disorder is the worst thing in small talents

Eine kleine Ordnung meines Innern fängt an sich herzustellen und nichts brauche ich mehr, denn Unordnung bei kleinen Fähigkeiten ist das Ärgste

Franz Kafka

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Preface

The following pages draw together, in the simplest and deftest way possible, the mathematical methods necessary for a first encounter with university-level science. The book is conceived of as an introduction to mathematics and logic for undergraduate students. It does not presuppose prior knowledge of mathematics or logic. However, it is strongly recommended that the mathematical chapters are studied consecutively. The chapters on logic can be read independently, beginning with propositional logic and ending with predicate logic. The section on mathematics is mainly organized according to subtle variations of Pythagoras' theorem across several topics such as analytic geometry, trigonometry, and calculus. The section on logic follows the traditional path of a transition from propositional to predicate logic, emphasizing the mechanical aspects of classical logical calculation.

The poetics of this textbook was inspired by Alistair C. Crombie and Ian Hacking's styles of scientific reasoning. I have tried to implement a mathematical style of reasoning that does not necessarily require long detailed explanations, using natural language. My assumption is that only a few elements of mathematical and logical reasoning are truly necessary to develop basic understanding. Students interested in expanding their knowledge further should consult the bibliography at the end of the book.

General Objectives of the Teaching Material

The principal objective of the book is to provide support for mathematical thinking and logic. It is geared towards introducing beginning students to mathematical and logical tools, developing their capacities for analytical thought and independent learning; and preparing them to confront the complexities of scientific knowledge.

Learning Objectives for the Development of Skills and Competencies

It is hoped that the student will: develop their capacities for geometric and algebraic reasoning and mathematical calculus; be able to apply the knowledge of mathematical logic to frame and resolve abstract problems; and improve their capacity for consistent argumentation.

On the Development of the Book and of the Author

This book finds its distant origins in a philosophy of science course delivered by Professor Javier Moscoso at the University of Murcia between 2002 and 2003. It is thanks to his expert guidance, as well as the invaluable support of my friend Mario Marín Marín, that I first discovered my interest in science and physics, in particular. Indeed, the effect was such that my intention had been that, upon completion of my undergraduate degree in philosophy, I would move on to study physics at the University of Strasbourg. However, an Esquerdo Foundation pre-doctoral scholarship at the Residencia de Estudiantes led to the postponement of this plan. During my years as a doctoral candidate at the Universidad Autónoma de Madrid, I had the opportunity to attend a number of mathematics courses at the Facultad de Ciencias Físicas of the Universidad Complutense (Madrid). Particularly important were the course in calculus given by the prematurely-deceased Joaquín Retamosa, another in linear algebra by José Ramón Peláez Sagredo, and two more in differential equations by Gabriel Álvarez Galindo and Pepe Aranda respectively. Gabriel Álvarez Galindo's classes were simply brilliant –both profound and engaging– and I will never be able to adequately thank my friend Abelardo Gil-Fournier for suggesting that I attend. As a result of its proximity to the Residencia de Estudiantes, I attempted somewhat to fill the gaps in my knowledge of statistics and probability through Camino González Fernández's course at the Escuela Técnica Superior of Industrial Engineers at the Universidad Politécnica de Madrid. After completing my PhD at the Universidad Autónoma de Madrid, and faced with a lack of professional opportunities, I returned to my earlier plan to start another undergraduate degree in the sciences, going so far as to enrol at the Centro Politécnico Superior at the Universidad de Zaragoza. This time, it was the award of a scholarship to undertake a second doctorate at the Scuola Internazionale di Alti Studi di Modena (Italy) that led me, once again, to abandon my dreams of a scientific Enlightenment. However, fate demanded that my teaching work at the Universidad Autónoma Metropolitana in Mexico City (UAM-Unidad Cuajimalpa) should begin with the courses "Introduction to Mathematical Thought" and "Logic" for humanities students. This book represents the fruits of both all past interest and my labors and experiences gained teaching in the Department of Humanities at the UAM-C. I am truly grateful to all of the people and institutions that have allowed me to pursue this alternative, belated and intermittently-developed vocation, and especially to the students in the Department of Humanities. This book has benefited from funding of the "History of Ecological Economics and Theory of Natural Capital" research project supported by the Mexican Program of Basic Scientific Research Conacyt/SEP (Reference 286529, 2018-2021). It is dedicated, with much admiration and affection, to the Quantum Geometry Group at the Instituto de Matemáticas (UNAM), and especially to Professor Micho Durdevich. Finally, I would like to thank Daniel S. Harper for the English translation. Any remaining mistakes or defects are, of course, my sole responsibility.

Part I

FUNDAMENTALS

Chapter I

Review of Algebra

1.1. Arithmetical Operations

$$a(b \pm c) = ab \pm ac \qquad \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

$$\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c} \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

Examples

$$2(2-3) = (2 \cdot 2) - (2 \cdot 3) = 4 - 6 = -2$$

$$\frac{5}{2} + \frac{3}{4} = \frac{5 \cdot 4 + 2 \cdot 3}{2 \cdot 4} = \frac{20+6}{8} = \frac{26}{8} = \frac{13}{4}$$

$$\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5} \qquad \frac{1/2}{1/3} = \frac{1 \cdot 3}{2 \cdot 1} = \frac{3}{2}$$

1.2. Exponentiation and Radical Expressions

$$x^m \cdot x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{m \cdot n} \qquad x^{-n} = \frac{1}{x^n} \text{ [i]}$$

[i] Proof: $\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n}$.

$$(x \cdot y)^n = x^n \cdot y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \text{ [ii]} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^m} \text{ [iii]}$$

$$x^0 = 1 \text{ [iv]} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Examples

$$2^2 \cdot 2^3 = 2^{2+3} = 2^5 \qquad \frac{2^3}{2^2} = 2^{3-2} = 2^1 = 2$$

$$(3^2)^3 = 3^{2 \cdot 3} = 3^6 \qquad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$(4 \cdot 3)^2 = 4^2 \cdot 3^2 \qquad \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2}$$

[ii] Proof by a change of variable: $x = a^{\frac{1}{n}}$; $x^n = \left(a^{\frac{1}{n}}\right)^n = a^{\frac{n}{n}} = a^1 = a$.

If $x^n = a$ then $\sqrt[n]{x^n} = \sqrt[n]{a}$, therefore $\sqrt[\cancel{n}]{x^{\cancel{n}}} = x = \sqrt[n]{a}$. In conclusion $a^{\frac{1}{n}} = \sqrt[n]{a}$.

[iii] Proof by a change of variable: $x = a^{\frac{m}{n}}$; $x^n = \left(a^{\frac{m}{n}}\right)^n = a^{\frac{m \cdot \cancel{n}}{\cancel{n}}} = a^m$.

If $x^n = a^m$ then $\sqrt[n]{x^n} = \sqrt[n]{a^m}$, so $\sqrt[\cancel{n}]{x^{\cancel{n}}} = x = \sqrt[n]{a^m}$. Therefore $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

[iv] Proof: $1 = \frac{x^n}{x^n} = x^{n-n} = x^0$.

$$5^{\frac{1}{3}} = \sqrt[3]{5} \qquad 2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}$$

$$2^0 = 1 \qquad \sqrt[3]{\frac{3}{2}} = \frac{\sqrt[3]{3}}{\sqrt[3]{2}}$$

1.3. Radical Expressions and Exponents

$$\sqrt[2]{A} = \sqrt{A}$$
$$\sqrt{A} = a \quad \text{if } a^2 = A$$
$$\sqrt[n]{A} = a \quad \text{if } a^n = A$$
$$a^{1/2} = \sqrt{a} \quad [\text{v}]$$
$$\sqrt{a^2} = a$$

Examples

$$\sqrt[2]{2} = \sqrt{2}$$
$$\sqrt{4} = 2 \;\rightarrow\; 2^2 = 4$$
$$\sqrt[3]{27} = 3 \;\rightarrow\; 3^3 = 27$$
$$5^{1/2} = \sqrt{5}$$
$$\sqrt{2^2} = 2$$

Properties of roots

$$\sqrt[n]{A^n} = A \qquad \sqrt[n]{A} \cdot \sqrt[n]{B} = \sqrt[n]{A \cdot B}$$

$$\sqrt[m]{\sqrt[n]{A}} = \sqrt[m \cdot n]{A} \qquad \frac{\sqrt[n]{A}}{\sqrt[n]{B}} = \sqrt[n]{\frac{A}{B}}$$

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[v] Proof by a change of variable: $a^{1/2} = x$; $x^2 = (a^{1/2})^2 = a^{2/2} = a^1 = a.$
$x^2 = a \rightarrow \sqrt{x^2} = \sqrt{a} \rightarrow x = \sqrt{a} = a^{1/2}.$

1.4. Key Algebraic Expressions

Square of a sum or a difference (the square of a binomial)

$$(a+b)^2 = a^2 + 2ab + b^2 \quad [\text{vi}]$$
$$(a-b)^2 = a^2 - 2ab + b^2 \quad [\text{vii}]$$

Cube of a sum or a difference

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Product of a sum and a difference

$$(a+b)(a-b) = a^2 - b^2 \quad [\text{viii}]$$

Examples

$$(2+3)^2 = (2^2 + 2\cdot 2\cdot 3 + 3^2) = (4+12+9) = 25$$
$$= (5)^2 = 25$$

$$(x-2)^2 = (x^2 - 2\cdot x\cdot 2 + 2^2) = (x^2 - 4x + 4) =$$
$$= (x-2)(x-2) = (x^2 - 2x - 2x + 2^2) = (x^2 - 4x + 4)$$

1.5. First-Order and Second-Degree Equations

$$ax + b = 0$$
$$ax^2 + bx + c = 0$$

[vi] Proof: $(a+b)^2 = (a+b)\cdot(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

[vii] Proof: $(a-b)^2 = (a-b)\cdot(a-b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$.

[viii] Proof: $(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$.