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On the representations in $\mathrm{GF}(3)^4$ of the Hadamard design $\mathcal{H}_{11}$

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Abstract

In this paper we study the representations of the 2-$(11, 5, 2)$ Hadamard design $\mathcal{H}_{11} = (\mathcal{P}, \mathcal{B})$ as a set of eleven points in the 4-dimensional vector space $\mathrm{GF}(3)^4$, under the conditions that the five points in each block sum up to zero, and $\dim\langle\mathcal{P}\rangle = 4$. We show that, up to linear automorphism, there exist precisely two distinct, linearly nonisomorphic representations, and, in either case, we characterize the family $\mathcal{S}$ of all the 5-subsets of $\mathcal{P}$ whose elements sum up to zero. In both cases, $\mathcal{S}$ properly contains the family of blocks $\mathcal{B}$, thereby showing that a previous result on the representations of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^5$ cannot be improved.

Key words: Block designs, Hadamard designs.
MSC: 05B05; 05B25; 51D20.

Contents

1 Introduction

The point set $\mathcal{P}$ of the 2-$(11, 5, 2)$ Hadamard design $\mathcal{H}_{11}$ can be represented as a set of eleven points in the 5-dimensional vector space $\mathrm{GF}(3)^5$ over $\mathrm{GF}(3)$, in such a way that the eleven blocks are precisely the only 5-subsets of $\mathcal{P}$ whose elements sum up to zero [2, Example 4.11]. This is a special case of a general property, which holds for all symmetric 2-designs not isomorphic to the trivial 2-$(v, v-1, v-2)$ design [2], for all affine 2-designs other than $\mathrm{AG}(2, 2)$ and for their complementary designs

[3], for the point-line designs of $\mathrm{PG}(d,2)$ and $\mathrm{AG}(d,3)$ (and no other Steiner triple systems) [2], and for the 2-designs introduced in [5].

In the case of the Hadamard design $\mathcal{H}_{11}$, one may ask if five coordinates are necessary for the validity of the above property, or whether it is possible to reduce the number of coordinates without losing the characterization of the blocks. In other words, the question is whether $\mathcal{H}_{11}$ can be embedded in $\mathrm{GF}(3)^4$, so that the eleven blocks are precisely the only 5-subsets of $\mathcal{P}$ whose elements sum up to zero.

In this paper we show that whenever the point set $\mathcal{P}$ of $\mathcal{H}_{11}$ is represented as a set of eleven points in $\mathrm{GF}(3)^4$, in such a way that the five points in each block sum up to zero, there always exists at least one 5-subset of $\mathcal{P}$ that is not a block, but whose elements sum up to zero as well, thereby showing that the representation of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^5$ given in [2] is the best possible.

We also show that, unlike in the case of the 5-dimensional representation, which is unique up to linear automorphism, there is no longer uniqueness in the case of 4-dimensional representations. Indeed, up to linear automorphism, $\mathcal{H}_{11}$ can be represented in two distinct, linearly nonisomorphic ways as a set of eleven points in $\mathrm{GF}(3)^4$, in such a way that the five points in each block sum up to zero, and that four points in $\mathcal{P}$ are linearly independent. Moreover, we show that the family $\mathcal{S}$ of all the 5-subsets of $\mathcal{P}$ whose elements sum up to zero has either 12 or 17 elements, the former case occurring if and only if there exists a point $P \neq 0$ such that $\mathcal{P}$ contains P and $-P$, and the latter case occurring if and only if $0 \in \mathcal{P}$. In the former case, the only 5-set in $\mathcal{S}$ that is not a block contains P, $-P$, and the three points in $\mathcal{P}$ not belonging to either of the two blocks through P and $-P$, whereas, in the latter case, the six 5-sets in $\mathcal{S}$ that are not blocks are the six sets of the form $\mathcal{P} \setminus (\{0\} \cup \mathfrak{b})$, with $\mathfrak{b}$ a block not containing 0. The same results can all be obtained in a more general setting and with a different approach [6].

In the case where $\mathcal{S}$ contains 12 elements, a characterization of the blocks of $\mathcal{H}_{11}$ can still be given, although in a weaker form: a 5-set in $\mathcal{S}$ is a block of $\mathcal{H}_{11}$ if and only if it intersects in two points each of the other 5-sets in $\mathcal{S}$, except at most one. Finally, in the case where $\mathcal{S}$ contains 17 elements, it is no longer possible to reconstruct the eleven blocks uniquely from $\mathcal{S}$, since there exist two distinct subfamilies of $\mathcal{S}$, each containing eleven elements, that can both be taken as the family of blocks of $\mathcal{H}_{11}$.

Moreover, we give an explicit representation of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$, with $P, -P \in \mathcal{P}$, for some $P \neq 0$, and $\dim\langle\mathcal{P}\rangle = 4$, and show that the group of the automorphisms of $\mathcal{H}_{11}$ that fix the pair $\{P, -P\}$ is isomorphic to the subgroup of $\mathrm{GL}_4(3)$ consisting of the matrices that permute the eleven points in $\mathcal{P}$. Such a group, in turn, is isomorphic to the dihedral group of order 12, of which we construct the two standard generators in $\mathrm{GL}_4(3)$.

Finally, we give an explicit representation of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$ also in the case where $0 \in \mathcal{P}$ and $\dim\langle\mathcal{P}\rangle = 4$, and define two matrices in $\mathrm{GL}_4(3)$ such that the group of matrices that they generate is isomorphic to the alternating group A_5 and coincides with the group of all the automorphisms of $\mathcal{H}_{11}$ that fix the point 0. Conversely, given the standard generators M and N of A_5 in its matrix representation of dimension 4 over GF(3) (see [9]), we construct a representation of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$, with $0 \in \mathcal{P}$ and $\dim\langle\mathcal{P}\rangle = 4$, such that the group of matrices generated by M and N coincides precisely with the group of all the automorphisms of $\mathcal{H}_{11}$ that fix the point 0.

2 Preliminaries

The 2-$(11, 5, 2)$ Hadamard design $\mathcal{H}_{11}$ is a pair $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set with eleven elements, called *points*, and $\mathcal{B}$ is a family of eleven 5-subsets of $\mathcal{P}$, called *blocks*, with the property that any two distinct blocks intersect in precisely two points [1, 4]. Equivalently, any two distinct points are contained in exactly two common blocks. It follows that every point belongs to precisely five blocks, and that for any two blocks there exist precisely three points that do not belong to either of the two blocks; conversely, for any two points there exist precisely three blocks that do not contain either of the two points.

The 2-$(11, 5, 2)$ Hadamard design is unique [8]: if $\mathcal{P} = \{P_1, P_2, \ldots, P_{11}\}$, then, up to permutation of the points, the eleven blocks $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_{11}$ of $\mathcal{H}_{11}$ can be taken as follows (see, for instance, [2, Example 2.3, Remark 4.12]).

$$\begin{aligned}
\mathfrak{b}_1 &= \{P_1, P_2, P_3, P_4, P_5\}\\
\mathfrak{b}_2 &= \{P_1, P_2, P_6, P_7, P_8\}\\
\mathfrak{b}_3 &= \{P_1, P_3, P_6, P_9, P_{10}\}\\
\mathfrak{b}_4 &= \{P_1, P_4, P_7, P_9, P_{11}\}\\
\mathfrak{b}_5 &= \{P_1, P_5, P_8, P_{10}, P_{11}\}\\
\mathfrak{b}_6 &= \{P_2, P_3, P_7, P_{10}, P_{11}\}\\
\mathfrak{b}_7 &= \{P_2, P_4, P_8, P_9, P_{10}\}\\
\mathfrak{b}_8 &= \{P_2, P_5, P_6, P_9, P_{11}\}\\
\mathfrak{b}_9 &= \{P_3, P_4, P_6, P_8, P_{11}\}\\
\mathfrak{b}_{10} &= \{P_3, P_5, P_7, P_8, P_9\}\\
\mathfrak{b}_{11} &= \{P_4, P_5, P_6, P_7, P_{10}\}.
\end{aligned} \tag{1}$$

Hereinafter, we will always represent $\mathcal{P}$ as a set of eleven (distinct) points in $\mathrm{GF}(3)^4$, in such a way that the five points in each of the blocks defined in (1) sum up to zero. Accordingly, the points $P_1, P_2, \ldots, P_{11}$ in $\mathcal{P}$ will satisfy the following equalities.

$$P_1 + P_2 + P_3 + P_4 + P_5 = 0 \tag{2}$$
$$P_1 + P_2 + P_6 + P_7 + P_8 = 0 \tag{3}$$
$$P_1 + P_3 + P_6 + P_9 + P_{10} = 0 \tag{4}$$
$$P_1 + P_4 + P_7 + P_9 + P_{11} = 0 \tag{5}$$
$$P_1 + P_5 + P_8 + P_{10} + P_{11} = 0 \tag{6}$$
$$P_2 + P_3 + P_7 + P_{10} + P_{11} = 0 \tag{7}$$

$$\begin{aligned} P_2 + P_4 + P_8 + P_9 + P_{10} &= 0 \qquad (8)\\ P_2 + P_5 + P_6 + P_9 + P_{11} &= 0 \qquad (9)\\ P_3 + P_4 + P_6 + P_8 + P_{11} &= 0 \qquad (10)\\ P_3 + P_5 + P_7 + P_8 + P_9 &= 0 \qquad (11)\\ P_4 + P_5 + P_6 + P_7 + P_{10} &= 0. \qquad (12) \end{aligned}$$

If this is the case, then we say that the pair $(\mathcal{P}, \mathcal{B})$ gives a *representation* of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$.

2.1. Remark: An elementary but essential fact about the representations of $\mathcal{H}_{11}$, which will be assumed throughout this paper, is that the relevant properties of $\mathcal{P}$ are invariant under linear isomorphism. Indeed, let $\mathcal{P} = \{P_1, P_2, \ldots, P_{11}\}$ be a subset of $\mathrm{GF}(3)^4$, such that the five points in each of the blocks defined in (1) sum up to zero, and let V be a vector space of dimension four over the ground field GF(3). If $\varphi : \mathrm{GF}(3)^4 \to V$ is a linear bijection, then $\sum_{P \in \mathfrak{b}} \varphi(P) = 0$ for each block $\mathfrak{b} \in \mathcal{B}$. Also, again by linearity, $X_1 + X_2 + X_3 + X_4 + X_5 = 0$ if and only if $\varphi(X_1) + \varphi(X_2) + \varphi(X_3) + \varphi(X_4) + \varphi(X_5) = 0$, for any 5-subset $\{X_1, X_2, X_3, X_4, X_5\} \subseteq \mathcal{P}$.

In the light of Remark 2.1, we will always have to specify, when we use the term *isomorphism*, if we mean design automorphism or linear automorphism. Indeed, as we recalled earlier, the 2-$(11, 5, 2)$ Hadamard design is unique [8], hence, in particular, any two representations $(\mathcal{P}_1, \mathcal{B}_1)$ and $(\mathcal{P}_2, \mathcal{B}_2)$ of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$ give two isomorphic designs, although, as we will see in the sequel of the paper, there may exist no linear automorphism φ of $\mathrm{GF}(3)^4$ such that $\varphi(\mathcal{P}_1) = \mathcal{P}_2$ and $\varphi(\mathcal{B}_1) = \mathcal{B}_2$. Thus it makes sense to give the following definition.

2.2. Definition: *Let $(\mathcal{P}_1, \mathcal{B}_1)$ and $(\mathcal{P}_2, \mathcal{B}_2)$ be two representations of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$. We say that they are linearly isomorphic if there exists a linear automorphism φ of $\mathrm{GF}(3)^4$ such that $\varphi(\mathcal{P}_1) = \mathcal{P}_2$, and $\varphi(\mathfrak{b})$ is a block in $\mathcal{B}_2$ for each block $\mathfrak{b}$ in $\mathcal{B}_1$.*

We are interested to investigate the (linear) isomorphism classes of the representations of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$, and study for each of them the family $\mathcal{S}$ of all the 5-subsets of $\mathcal{P}$ whose elements sum up to zero. The main motivation behind the present paper is to establish whether there exist cases where the family $\mathcal{S}$ reduces to just the family $\mathcal{B}$ of blocks, as it happens for the 5-dimensional representation.

3 The main theorems

Our first goal in this section is to show that if $(\mathcal{P}, \mathcal{B})$ is a representation of $\mathcal{H}_{11}$ in $\mathrm{GF}(3)^4$, then there always exists at least one 5-subset of $\mathcal{P}$ which is not a block, but whose elements sum up to zero.

3.1. Lemma: *Let* $(\mathcal{P}, \mathcal{B})$ *be a representation of* $\mathcal{H}_{11}$ *in* $\mathrm{GF}(3)^4$. *Then*

$$\sum_{X \in \mathcal{P}} X = 0.$$

Moreover, whenever P *and* Q *are two distinct points in* $\mathcal{P}$, *and* A, B, C *are the three points in* $\mathcal{P}$ *not belonging to either of the two blocks through* P *and* Q,

$$A + B + C = P + Q.$$

Proof. Since each point in $\mathcal{P}$ belongs to precisely 5 blocks, we may conclude, by a a double-counting argument, that $0 = \sum_{\mathfrak{b} \in \mathcal{B}} \sum_{X \in \mathfrak{b}} X = 5 \sum_{X \in \mathcal{P}} X$, whence our first claim follows.

Let P and Q be two distinct points in $\mathcal{P}$, let $\mathfrak{b}$ and $\mathfrak{b}'$ be the two blocks through P and Q, and let A, B, C be the three points in $\mathcal{P}$ not belonging to either $\mathfrak{b}$ or $\mathfrak{b}'$. Since $\mathcal{P} = \mathfrak{b} \cup \mathfrak{b}' \cup \{A, B, C\}$, and $\mathfrak{b} \cap \mathfrak{b}' = \{P, Q\}$, the equality $\sum_{X \in \mathcal{P}} X = 0$ may be rewritten as

$$\sum_{X \in \mathfrak{b}} X + \sum_{X \in \mathfrak{b}'} X - (P + Q) + \sum_{X \in \{A,B,C\}} X = 0,$$

whence $A + B + C = P + Q$, as claimed. $\square$

3.2. Lemma: *Let* $(\mathcal{P}, \mathcal{B})$ *be a representation of* $\mathcal{H}_{11}$ *in* $\mathrm{GF}(3)^4$. *If* $\mathcal{P}$ *contains either* 0 *or a pair of opposite elements* P *and* $-P$ $(P \neq 0)$, *then there exists at least one* 5-*subset of* $\mathcal{P}$ *which is not a block, but whose elements sum up to zero.*

Proof. If $0 = (0, 0, 0, 0) \in \mathcal{P}$, and if $\mathfrak{b}$ is a given block in $\mathcal{B}$ not containing 0, then $\mathfrak{s} = \mathcal{P} \setminus (\{0\} \cup \mathfrak{b})$ is a 5-subset of $\mathcal{P}$ and $\sum_{X \in \mathfrak{s}} X = \sum_{X \in \mathcal{P}} X - \left(0 + \sum_{X \in \mathfrak{b}} X\right) = 0 - 0 = 0$ by Lemma 3.1. Also, $\mathfrak{s}$ is not a block, as $\mathfrak{s} \cap \mathfrak{b} = \emptyset$.

If there exists $P \neq 0$ such that P and $-P$ both belong to $\mathcal{P}$, then let $\mathfrak{s}$ be the 5-subset of $\mathcal{P}$ defined by $\mathfrak{s} = \{P, -P, A, B, C\}$, where A, B, C are precisely the three points in $\mathcal{P}$ not belonging to either of the two blocks through P and $-P$. Then, by Lemma 3.1, $A + B + C = P + (-P) = 0$, thus the five elements of $\mathfrak{s}$ sum up to zero. Finally, $\mathfrak{s}$ is not a block, else there would exist three distinct blocks through P and $-P$. $\square$

3.3. Lemma: *Let $(\mathcal{P}, \mathcal{B})$ be a representation of $\mathcal{H}_{11}$ in* $\mathrm{GF}(3)^4$. *Then there exists at least one block* $\mathfrak{b}$ *in* $\mathcal{B}$ *such that the linear span* $\langle\mathfrak{b}\rangle$ *of the elements of* $\mathfrak{b}$ *in* $\mathrm{GF}(3)^4$ *has dimension* 3.

Proof. The linear span $\langle\mathcal{P}\rangle$ of $\mathcal{P}$ in $\mathrm{GF}(3)^4$ must necessarily have dimension three or four, since $\mathcal{P} = \{P_1, P_2, \ldots, P_{11}\}$ has eleven elements and the ground field $\mathrm{GF}(3)$ has only three elements. Let P and Q be two distinct nonzero points in $\mathcal{P}$, with $Q \neq -P$. Equivalently, P and Q are linearly independent elements of $\mathcal{P}$. Up to permutation of the points, we may assume that $\{P, Q\} = \{P_1, P_2\}$ and that the two blocks through P and Q are the blocks $\mathfrak{b}_1$ and $\mathfrak{b}_2$ defined in (1). As P_9, P_{10}, and P_{11} are linear combinations of $P_1, P_2, \ldots, P_8$ by the equalities (10), (11), and (12), we may conclude that the linear span of $\mathfrak{b}_1 \cup \mathfrak{b}_2 = \{P_1, P_2, \ldots, P_8\}$ is all of $\langle\mathcal{P}\rangle$ and hence has dimension at least 3. Therefore either $\langle\mathfrak{b}_1\rangle$ or $\langle\mathfrak{b}_2\rangle$ must have dimension greater than two, else P_1 and P_2 would generate all of $\langle\mathfrak{b}_1 \cup \mathfrak{b}_2\rangle = \langle\mathcal{P}\rangle$, against the fact that $\dim\langle\mathcal{P}\rangle = 3$.

If either $\langle\mathfrak{b}_1\rangle$ or $\langle\mathfrak{b}_2\rangle$ has dimension 3, then there is nothing else to prove, else one of the two, say $\langle\mathfrak{b}_1\rangle$, has dimension 4. In the latter case we can assume, up to linear isomorphism (see Remark 2.1), and because of the equality (2), that

$$\begin{aligned} P_1 &= (1,0,0,0) \\ P_2 &= (0,1,0,0) \\ P_3 &= (0,0,1,0) \\ P_4 &= (0,0,0,1) \\ P_5 &= (2,2,2,2). \end{aligned}$$

If now

$$P_8 = (a,b,c,d),$$

then the coordinates of the remaining five elements of $\mathcal{P}$ are uniquely determined. Indeed, since P_1, P_2, P_{10} are precisely the three points in $\mathcal{P}$ not belonging to the two blocks $\mathfrak{b}_9$ and $\mathfrak{b}_{10}$ through P_3 and P_8, we can conclude, by Lemma 3.1, that $P_1 + P_2 + P_{10} = P_3 + P_8$, whence $P_{10} = P_3 + P_8 - P_1 - P_2$, that is,

$$P_{10} = (a+2, b+2, c+1, d).$$

By the equality (6), $P_1 + P_5 + P_8 + P_{10} + P_{11} = 0$, that is, $P_{11} = -P_1 - P_5 - P_8 - P_{10}$, thus

$$P_{11} = (a+1, b+2, c, d+1).$$

Similarly, one obtains, by means of the equalities (10), (3), and (8), that

$$\begin{aligned} P_6 &= (a+2, b+1, c+2, d+1) \\ P_7 &= (a, b+1, c+1, d+2) \\ P_9 &= (a+1, b, c+2, d+2), \end{aligned}$$

respectively.

We can now finally prove that there exists a block $\mathfrak{b}$ in $\mathcal{B}$, necessarily different from $\mathfrak{b}_1$, such that $\dim\langle\mathfrak{b}\rangle = 3$.

By the equality (3), the point P_7 is a linear combination of P_1, P_2, P_6, and P_8, hence $\dim\langle\mathfrak{b}_2\rangle = 3$ if and only if $\dim\langle\{P_1, P_2, P_6, P_8\}\rangle = 3$. Let A be the 4x4 matrix in $M_4(\mathrm{GF}(3))$ whose columns are the column vectors P_1, P_2, P_8, and P_6, in this order, that is,

$$A = \begin{pmatrix} 1 & 0 & a & a+2 \\ 0 & 1 & b & b+1 \\ 0 & 0 & c & c+2 \\ 0 & 0 & d & d+1 \end{pmatrix}.$$

Since the column vectors $(c, d)'$ and $(c+2, d+1)'$ cannot be both equal to $(0, 0)'$ in $\mathrm{GF}(3)^2$, the dimension of $\langle\{P_1, P_2, P_6, P_8\}\rangle$ in $\mathrm{GF}(3)^4$ is equal to either 3 or 4, the former case occurring if and only if $\det(A) = 0$. Therefore,

$$\dim\langle\mathfrak{b}_2\rangle = 3 \iff c + d = 0. \tag{13}$$

One can easily show, by a similar argument, that

$$\dim\langle\mathfrak{b}_3\rangle = 3 \iff b + d + 2 = 0 \tag{14}$$
$$\dim\langle\mathfrak{b}_4\rangle = 3 \iff b + c + 2 = 0 \tag{15}$$
$$\dim\langle\mathfrak{b}_5\rangle = 3 \iff b + c + d = 0 \tag{16}$$
$$\dim\langle\mathfrak{b}_6\rangle = 3 \iff a + d + 2 = 0 \tag{17}$$
$$\dim\langle\mathfrak{b}_7\rangle = 3 \iff a + c = 0 \tag{18}$$
$$\dim\langle\mathfrak{b}_8\rangle = 3 \iff a + c + d + 2 = 0 \tag{19}$$
$$\dim\langle\mathfrak{b}_9\rangle = 3 \iff a + b = 0 \tag{20}$$
$$\dim\langle\mathfrak{b}_{10}\rangle = 3 \iff a + b + d = 0 \tag{21}$$
$$\dim\langle\mathfrak{b}_{11}\rangle = 3 \iff a + b + c + 2 = 0. \tag{22}$$

One can now show that at least one of the conditions $(13), \ldots, (22)$ is satisfied, for any instance of (a, b, c, d) in $\mathrm{GF}(3)^4$. One can proceed, for instance, as follows. If $a + b = 0$, then (20) is satisfied. Let us next consider the case where $a + b = 1$. If $c = 0$, then (22) is satisfied, whereas, if $d = 2$, then (21) is satisfied. Therefore, for $a + b = 1$, it suffices to consider the cases where $c = 1, 2$, and $d = 0, 1$. If $c = 1$ and $b = 0$ (respectively, $b = 2$), then (15) (respectively, (18)) is satisfied, whereas, if $c = 1$, $b = 1$ and $d = 0$ (respectively, $d = 1$), then (14) (respectively, (16)) is satisfied. The remaining cases where $a + b = 1$ and $c = 2$, and all the cases where $a + b = 2$ are disposed of similarly.

The proof is now complete. $\square$

We can now state the main theorem of this paper.