TEMPUS PECUNIA EST

COLLANA DI MATEMATICA PER LE SCIENZE ECONOMICHE
FINANZIARIE E AZIENDALI

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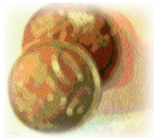

Al suo livello più profondo la realtà è la matematica della natura.

Pitagora

Questa collana nasce dall'esigenza di offrire al lettore dei trattati che aiutino la comprensione e l'approfondimento dei concetti matematici che caratterizzano le discipline dei corsi proposti nelle facoltà di Scienze economiche, finanziarie e aziendali.

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Mathematical Tools in Economic and Financial Models

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Preface

This text was written for advanced undergraduate and beginners graduated students, as well as researchers who want to deepen their knowledge in some mathematical methods very useful in the recent economic and financial literature.

It has been made through the experience of lecturing conducted teaching for many years in PHD advanced courses in Quantitative Methods at the University of Cagliari, Department of Economics and Business, Italy. So, the text follows the style of the lectures. It deals with deterministic and stochastic differential equations. Then, through the applications and the problems proposed, readers have the chance to strengthen and deepen their preparation the mathematical topics proposed.

Cagliari, May 2019

Chapter I

Differential Equations

1.1. Ordinary differential equations

1.1.1. *Definition*

A first order ordinary differential equation (O.D.E.) is a mathematical expression within which we find an **unknown function** $y(x)$ and its **first derivative** $y'(x)$.

Definition

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a ***differential equation****.*

The **order** of the differential equation is given by the maximum derivative which appears in the unknown function: if the highest-order derivative is the first, we have a **first order** differential equation (and so on, if the highest-order derivative is the second, we have a **second order** differential equation). For instance:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - y = 0$$

is a second-order ordinary differential equation, because we have the term $\frac{d^2y}{dx^2}$. function y is defined in an open interval from a to b in $\mathbb{R}$:

$$y \ :(a, b) \in \mathbb{R}$$

1.1.2. *First order differential equations*

Let's consider:

$$F\big(x, y(x), y'(x)\big) = 0$$

where:

- $y(x)$ = unknown function
- $y'(x)$ = first derivative
- x = independent variable

The O.D.E it's a first order differential equation (in its implicit form). In the **explicit form** we can find at the first side the first derivative (y'):

$$\frac{dy}{dx} = f\big(x, y(x)\big) \quad (1.1)$$

here f is the known function.

Some other examples of first order differential equation can be:

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = I(t)$$

A differential equation is said to be **linear** if the unknown function is a first order function:

$$\frac{dy}{dx} + x^2 y(x) = e^x$$

in this case, although x^2 is not linear, this differential equation is however linear: it's the degree of the unknown function $y(x)$ which determines the linearity.

A linear differential equation can be:

i) **homogeneous**, if the second term is zero:

$$\frac{dy}{dx} + y(x) = 0$$

ii) **non-homogeneous**, if we have a variable in the second term, like e^x. which depends on x. which itself depends on the unknown function:

$$\frac{dy}{dx} + y(x) = e^x$$

A **nonlinear** ordinary differential equation is just one that isn't linear. Nonlinear functions of the dependent variable or its derivatives, such as *sin y*, cannot appear in a linear equation. Therefore:

$$(1 - y)y' + 2y = e^x \text{ (coefficient depends on y)}$$

$$\frac{d^2y}{dx^2} + sin\, y = 0$$

$$\frac{d^4y}{dx^4} + y^2 = 0$$

are examples of nonlinear first-order, second- order, and fourth-order ordinary differential equations, respectively.

We can observe the differences between numerical and differential equations, in both, we have to find an unknown quantity:

- in numerical equation, such as $ax + b = 0$ the unknown quantity, $x = \frac{-b}{a}$. is a real number;
- in the differential equations we are looking for an unknown function $y(x)$. and not a number.

1.1.3. *Solution of a first order ODE*

Solving a differential equation means to find the function $g(x)$ (called **solution** or integral) that makes the expression identically satisfied. A function $g(x)$ is a solution when, substituted into ODE (1.1), it reduces to an identity in a certain open interval (a, b) in $\mathbb{R}$. Generally, we can find the solution by integration.

1.1.4. *The separation of variables method*

Let $y^{'} = f(x)g(y)$ be an ordinary differential equation as a product of two functions: the function f depends on the variable x and the function g depends on the variable y. The "separation of variables" method separates the two variables y and x placing them in different sides of the equation; each side is then integrated:

$$y' = f(x)g(y)$$

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

Definition

A first-order differential equation of the form $\left(\frac{dy}{dx}\right) = f(x)g(y)$ *is called separable.*

For example, let's consider one of the easiest differential equations:

$$y' = y$$

Now the question is: what is the function which is equal to its derivative? It is the exponential. Generally speaking, we can assume that if the differential equation is linear, its solution is an **exponential**.

We can use the separation of variables method to find the solution: we can write the explicit form like a product at the second side ($y \cdot 1$)

$$y' = y \cdot 1$$

y depends on x. we have a derivative on the left-hand side:

$$\frac{dy}{dx} = y$$

we separate the differentials:

$$\frac{dy}{y} = dx$$

and we proceed with integration by parts.

Integration is the inverse operation of the derivative. If we are looking for the unknown function, we have to integrate:

$$\int \frac{dy}{y} = \int dx$$

At the first side we have:

$$\int \frac{y'}{y} = \int dx$$

If we have a function in the denominator and its derivative at numerator, that is by definition a **growth rate**, which is the derivative of a logarithm itself. The primitive is the function $ln\, y$:

$$ln\, y = x + C$$

Ça va sans dire: when we integrate, we must not forget to add a constant of integration C. Of couse, here we are not interested in the $ln\, y$. but we are looking for y. In order to discover it, we apply the exponential (which is the inverse function of logarithm) to both sides:

$$e^{\ln y} = e^{(x+C)}$$

and we get:

$$y = e^{(x+C)}$$

for the properties of powers:

$$y = e^x e^C$$

Generally, we can write e^C in this more elegant form:

$$y = Ce^x$$

This is the general solution of the first order linear differential equation $y' = y$. Let $C = 1$ we have $y = e^x$ (see Fig 1):

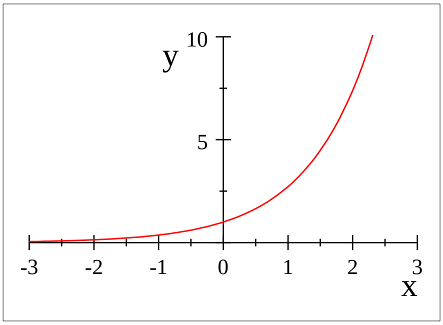

Figure 1.1.

1.1.5. *Examples of non-linear differential equations*

Here we have two examples of nonlinear differential equations:

$$y' = xy^2$$

$$y' = \sqrt{y}$$

In this case the link between y and its first derivative y' is not linear and certainly we won't have an exponential solution.

Example 1:

$$y' = xy^2$$

We apply the separation of variables method:

$$\frac{dy}{dx} = xy^2$$

$$\frac{dy}{y^2} = x \cdot dx$$

$$\int \frac{dy}{y^2} = \int x \cdot dx$$

$$\int y^{-2} dy = \int x \cdot dx$$

$$\frac{y^{-2+1}}{-2+1} = \frac{x^{1+1}}{1+1}$$

$$\frac{y^{-1}}{-1} = \frac{x^2}{2}$$

$$\frac{-1}{y} = \frac{x^2}{2} + C$$

We calculate the least common multiple on the right-hand side:

$$\frac{1}{y} = \frac{x^2 + 2C}{2}$$

The solution is:

$$y = \frac{2}{x^2 + 2C}$$

Example 2:

$$y' = \sqrt{y} = y^{\frac{1}{2}}$$

$$\frac{dy}{dx} = y^{1/2}$$

$$\frac{dy}{y^{1/2}} = dx$$

$$\int \frac{dy}{y^{1/2}} = \int dx$$

$$\int y^{-1/2} dy = \int dx$$

$$\frac{y^{-1/2+1}}{-1/2+1} = x + C$$

$$2y^{1/2} = x + C$$

$$y^{1/2} = \frac{x+C}{2}$$

$$y(x) = \left(\frac{x+C}{2}\right)^2$$

1.1.6. *Particular solution*

We have a particular solution when we assign a particular value to the solution. Let's define such a particular solution or integral of the differential equation $F(x, y, y') = 0$ each function $y = F(x)$ obtained by assigning particular values to the arbitrary constant $y = \varphi(x)$.

For example:

$$y' - x^2 = 0$$

given the initial conditions $P\left(2;\frac{2}{3}\right)$:

$$y' = x^2$$

$$\frac{dy}{dx} = x^2$$

$$dy = x^2 \cdot dx$$

$$\int dy = \int x^2 \cdot dx$$

$$y = \frac{x^3}{3} + C$$

by substituting the initial conditions:

$$\frac{2}{3} = \frac{2^3}{3} + C$$

we get C:

$$C = -2$$

So the particular solution is:

$$y = \frac{x^3}{3} - 2$$

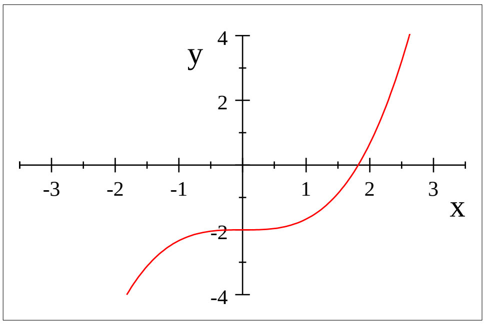

Figure 1.2.

$C = -2$ is exactly the point where the function intersects the vertical axes.

1.1.7. *Singular integral*

A singular integral is the solution that we cannot obtain by assigning a value to the constant C.

For instance:

$$y' = 2\sqrt{y}$$

$$\frac{dy}{dx} = 2\sqrt{y}$$

$$\frac{dy}{2\sqrt{2}} = dx$$

$$\frac{1}{2}(y)^{-\frac{1}{2}}dy = dx$$

$$\frac{1}{2}\int (y)^{-\frac{1}{2}}dy = \int dx$$

$$\frac{1}{2} \cdot \frac{y^{\frac{1}{2}}}{\frac{1}{2}} = x + C$$

$$y^{\frac{1}{2}} = x + C$$

$$y = (x + C)^2$$

Notice that $y = 0$ is a solution but this solution cannot be obtained by assigning a value to C from the general solution.