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On the conductor of algebraic varieties with multilinear tangent cones at isolated singularities

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Abstract

Let A be the local ring, with maximal ideal $\mathfrak{m}$, of an affine algebraic variety $V \subset \mathbb{A}_k^{r+1}$ (over an algebraically closed field k of characteristic zero) with dimension $d+1$ and regular normalization $\overline{A}$. Let P be an isolated singular point of V of multiplicity e. Assume that the projectivized tangent cone W of V at P consists of a union of linear varieties $L_i, i = 1, ..., e$ in generic position that is the Hilbert function of W is $H_W(n) = min\{\binom{n+r}{r}, e\binom{n+d}{d}\}$, for any n (i.e. maximal). Assume that these varieties are also in generic $e-1$ position that is the Hilbert function of $W - L_i$ is maximal for any $i = 1, ..., e$. Set $s = Min\{n \in \mathbb{N} | (e-1)\binom{n+d}{d} < \binom{n+r}{r}\}$. In this paper we prove that the conductor $\mathfrak{b}$ of A in $\overline{A}$ is $\mathfrak{m}^s$ if and only if $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ (note that the condition $e = \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ holds in a few sporadic cases). This extends to all varieties of this type the results of [1], [7] and of [9] for curves and surfaces.

A.M.S. Subject Classification : 14Q15

Key Words: Algebraic varieties, conductor, linear varieties, tangent cones

Introduction

Let A be the local ring, with maximal ideal $\mathfrak{m}$, of an affine equidimensional algebraic variety $V \subset \mathbb{A}_k^{r+1}$ (over an algebraically closed field k of characteristic zero) of dimension $d+1$ and P be a point of V of multiplicity e. P is said to be a singular point (or a singularity) of V if $e > 1$. Let $\overline{A}$ be the normalization of A. The (ideal) conductor $\mathfrak{b}$ of A in $\overline{A}$ is defined as $\mathfrak{b} = Ann_A(\overline{A}/A)$. It is well known that P is a singular point if and only $\mathfrak{b}$ is a proper ideal. The conductor at a singular point has a tight connection with the type of singularity of P and it is has a strict relation with other topics of algebraic geometry like the theory of adjoints and the theory of seminormal varieties. In general very few is known on the properties of the conductor but, in the case of an ordinary singularity of a plane curve (that is a singularity that has e distinct tangents or equivalently whose projective tangent cone $Proj(G(A))$ has e distinct points), as a consequence of a theorem of Nortcott and Matlis [4], one gets that $\mathfrak{b} = \mathfrak{m}^{e-1}$. The most simple case is a node for which $\mathfrak{b} = \mathfrak{m}$. This result was generalized in [7], Proposition 3.5 and Theorem 4.4, to space curves V for which P is an ordinary singularity with the e points of the projective tangent cone in generic $e-1, e$, position by proving that $\mathfrak{b} = \mathfrak{m}^s$, where $s = Min\{n \in \mathbb{N} | e \leq \binom{n+r}{r}\}$. After that in [9] (see also [1]) it was proved that the conductor of a surface, with regular normalization and whose projective tangent cone at an isolated singularity consists of e lines in generic $e-1, e$ position, is $\mathfrak{b} = \mathfrak{m}^s$ (where $s = Min\{n \in \mathbb{N} | (e-1)(n+1) < \binom{n+r}{r}\}$) if and only if $e \neq \lfloor \binom{s+r}{r}/(s+1) \rfloor + 1$. In this paper we unify all these results in the following result valid for the conductor of any variety with regular normalization at a singular isolated point P with projectivized tangent consisting of a union of e linear varieties in generic $e-1, e$ position by proving that, if $s = Min\{n \in \mathbb{N} | (e-1)\binom{n+d}{d} < \binom{n+r}{r}\}$, the conductor $\mathfrak{b}$ of A in $\overline{A}$ is $\mathfrak{m}^s$ if and only if $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$

If S is a semilocal ring,with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_e$ by $G(S)$ we denote the associated graded ring $\bigoplus_{n \geq 0} (\mathfrak{J}^n / \mathfrak{J}^{n+1})$ with respect to the Jacobson radical ideal $\mathfrak{J} = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_e$ of S. If $x \in S$, $x \neq 0$,$x \in \mathfrak{J}^n - \mathfrak{J}^{n+1}$, $n \in \mathbf{N}$ we say that x has degree n and the image $x^* \in \mathfrak{J}^n / \mathfrak{J}^{n+1}$, of x in $G(S)$ is said to be the initial form of x. If $\mathfrak{a}$ is an ideal of S, by $G(\mathfrak{a})$ we denote the ideal of $G(S)$ generated by all the initial forms of the elements of $\mathfrak{a}$.

With $(A, \mathfrak{m})$ we denote the local ring with maximal ideal $\mathfrak{m}$. $k = A/\mathfrak{m}$ is the residue field of A. $H(A, n) = dim_k(\mathfrak{m}^n / \mathfrak{m}^{n+1})$, $n \in \mathbf{N}$, denotes the Hilbert function of A and $e(A)$ is the multiplicity of A at $\mathfrak{m}$. The embedding dimension $emdim(A)$ of A is given by $H(n, 1)$.

If $R = \bigoplus_{n \geq 0} R_n$ is a standard graded finitely generated algebra over a field k, of maximal homogeneous ideal $\mathfrak{n}$, $H(R, n) = dim_k(R_\mathfrak{n}) = H(R_\mathfrak{n}, n)$ denotes the Hilbert function of R and $emdim(R) = H(R, 1) = emdim(R_\mathfrak{n})$ the embedding dimension of R. The multiplicity of R is $e(R) = e(R_\mathfrak{n})$. One has $e(A) = e(G(A))$ and $emdim(A) = emdim(G(A))$.

If B is any ring $\overline{B}$ denotes the normalization of B. If A is a subring of B $Ann_A(B/A)) = \{x \in A \mid Bx \subset A\}$ is the conductor of A in B (that is the largest ideal of A and B). In the following for conductor of B we mean the conductor of B in its normalization $\overline{B}$.