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Nikolaj Sergeevič Černikov (1955 – 2018)

Nikolaj Sergeevič Černikov passed away at the age of 63 on July 27th, 2018. He was born on May 12th, 1955 in Perm (Russia). In 1964, Černikov's family moved to Kiev, and in 1972 he enrolled the *Mechanics and Mathematics Department* of *Taras Shevchenko University*, where he graduated in 1977. Since then, his scentific life is connected with the *Institute of Mathematics* of the *National Academy of Sciences* of Ukraine. Nikolaj was still a student when he started doing research in group theory, achieving significant results under the guidance of his father Sergej N. Černikov, and he was an active member of the *Kiev Group Theory Seminar* since 1975. In 1978, he defended his Ph.D. dissertation, prepared under the supervision of Vladimir P. Šunkov. Nikolaj was still a student when he started doing research in group theory, achieving significant results under the guidance of his father Sergej N. Černikov, and he was an active member of the *Kiev Group Theory Seminar* since 1975. Nikolaj Černikov published more than eighty papers devoted to relevant questions of the theory of groups, mainly working on groups with generalized conditions of minimality and solubility, factorizations of groups and linear groups. He wrote a monograph on the theory of factorized groups, and among many other results in this topic he proved that any group, which is the product of two subgroups that are finite over the centre, contains a soluble subgroup of finite index. His work has made him a renowned expert in the field of group theory.

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Fox Derivatives: a Unique Connection Between Group Presentations and the Integral Group Ring *

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Dedicated to Iraj and Mehri

Abstract

We give a criterion to determine whether generators can be removed from a finite presentation via Tietze transformations. We prove that for a generator in a presentation $\langle X|R\rangle$ to be removable, there must exist a word in the normal closure of relators, $\overline{R}$, whose Fox derivative is an invertible element in $\mathbb{Z}G$. Furthermore, in this case all elements of $\mathbb{Z}G$ can be written as the derivative of words in $\overline{R}$, with respect to the removable generator. We further discuss the application of this result on the theory units of group rings.

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Keywords: group presentation; Fox derivative; unit of the integral group ring

1 Introduction

Motivated by Knot Theory, Ralph H. Fox defined what we now call the Fox Derivative for free groups and extended the definition to all groups via group presentations. As a result, the Jacobian matrix and the elementary

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ideals, as defined by Fox, reveal critical information about both the group's isomorphism type and its presentation. Most of these results are described in [1] and [2]. In particular, in [2] it is demonstrated that Tietze Transformations, when applied to the Jacobian, give an equivalent matrix. The work here, as described in the Preliminaries, is inspired by the cases where the inverse of this correspondence fails. Specifically, if the presentation has a superfluous generator, then the Jacobian matrix has a column and row of certain description. Our question is whether a presentation with a Jacobian satisfying that description, must have a generator that is removable in the following sense.

Definition 1.1 *Given a presentation* $\langle X|R\rangle$ *for group* G *and* $y \in X$*, we say that generator* y *is removable, if there exists a word* $w \in F(X \setminus \{y\})$ *such that* $y^{-1}w \in \overline{R}$*.*

Although the correspondence of presentations and equivalent Jacobians fails, our main result shows that if we find a column and row of the mentioned description, a generator is indeed removable, and superfluous in the presentation. Specifically, this happens if and only if the Fox derivative of a word in the normal closure of R is invertible.

Theorem 1.2 *Let* $\langle X|R\rangle$ *be a presentation for group* G *and* $y \in X$*. Then, there exists a word* $f \in \overline{R}$ *such that*

$$\Psi\frac{\partial f}{\partial y}$$

is an invertible element in $\mathbb{Z}G$ *if and only if the generator* y *is removable in this presentation.*

Here Ψ is the map evaluating words in $F(X)$ as group elements in G, when linearly extended to $\mathbb{Z}F(X)$, and

$$\frac{\partial}{\partial y} : F(X) \to \mathbb{Z}F(X)$$

is the Fox derivative with respect to y.

In Section 4, we discuss the potential applications of the main theorem to the theory of group rings. In particular, Corollary 4.2 states if y is removable, all non-trivial units in $\mathbb{Z}G$ can be written as the derivatives of words in the normal closure of R. Furthermore, in the proof of the main result, we further describe such words. Hence, we discuss the potential use of the Theorem for finding non-trivial units in the integral group ring.

2 Preliminaries

First we recall the theory of group presentations: given a set X and a subset R of the free group $F(X)$, we say $\langle X|R\rangle$ is a *presentation* for group G, if there exists a surjective homomorphism

$$\psi : F(X) \to G$$

with $\mathrm{Ker}(\psi) = \overline{R}$, where $\overline{R}$ is the normal closure of the set R in $F(X)$. Elements of X and R are called *generators* and *relators*, respectively. Furthemore, we say $\langle X|R\rangle$ is a *finite presentation* if both X and R are finite sets. As a standard reference on presentations of groups we refer to [3].

We recall Tietze's four transformations on the presentation of a group from [3]:

(T1) Addition of relator: $X' = X, R' = R \cup \{r\}$ where $r \in \overline{R} \setminus R$.

(T2) Removal of a relator: $X' = X, R' = R \setminus \{r\}$ where $r \in R \cap \overline{R \setminus \{r\}}$.

(T3) Addition of a generator: $X' = X \cup \{y\}, R' = R \cup \{y^{-1}w\}$ where $y \notin X$ and $w \in F(X)$.

(T4) Removal of generator: $X' = X \setminus \{y\}, R' = R \setminus \{y^{-1}w\}$ where $y \in X$, $w \in F(X \setminus \{y\})$ and $y^{-1}w$ is the only word in R involving y.

where each transformation takes us from a presentation $\langle X|R\rangle$ to presentation $\langle X'|R'\rangle$ for the same group. Notice that given a finite presentation for group G, if generator y is removable as in Definition 1.1, one can first add the relator $y^{-1}w$ to the presentation by a (T1) transformation, and then substitute y by w in all other relators by a series of (T1) and (T2) transformations. This will not change the group isomorphism type. Consequently, $y^{-1}w$ will be the only remaining relator containing y and generator y can then be removed by (T4).

We now recall Fox's construction of derivatives on the free group $F(X)$ from [1]: for any $x \in X$, there exists a map

$$\frac{\partial}{\partial x} : F(X) \to \mathbb{Z}F(X)$$

with defining properties

$$\text{(a)} \quad \frac{\partial y}{\partial x} = \begin{cases} 0 & y \neq x \\ 1 & y = x \end{cases} \qquad\qquad \text{(b)} \quad \frac{\partial wv}{\partial x} = \frac{\partial w}{\partial x} v + \frac{\partial v}{\partial x}$$

for $y \in X$ and $w, v \in F(X)$. From the above axioms one can easily conclude that for any word

$$w = u_m x^{p_m} u_{m-1} \dots x^{p_1} u_0 \in F(X),$$

where $u_i \in F(X \setminus \{x\})$ and p_i are non-zero integers,

$$\frac{\partial w}{\partial x} = \sum_{i=1}^{m} \left(x^{p_i - 1} + \dots + x + 1 \right) u_{i-1} x^{p_{i-1}} \dots x^{p_1} u_0 \tag{2.1}$$

Another important corollary of (a) and (b) is the identity

$$\frac{\partial w^{-1}}{\partial x} = -\frac{\partial w}{\partial x} w^{-1} \tag{2.2}$$

holding for any word $w \in F(X)$.

Now recall Fox's extension of the derivative to the group ring $\mathbb{Z}G$. Let

$$\Psi : \mathbb{Z}F(X) \to \mathbb{Z}G$$

be the natural linear extension of the map ψ. In [2], Fox explores the results which arise when applying the composite map

$$\Psi \frac{\partial}{\partial x} : F(X) \to \mathbb{Z}G$$

on the relators of the presentation.

Notation

- From this point we will use notation $\simeq$ for equality in $\mathbb{Z}G$, and $=$ for equality in $\mathbb{Z}F$, to avoid confusion between two words being equal and two words representing the same group element. Moreover, when we say a word w has *derivative* equal to $\alpha \in \mathbb{Z}G$, we mean $\Psi\frac{\partial w}{\partial x} \simeq \alpha$.
- Given a group presentation $\langle X|R\rangle$ with $y \in X$, we say y *appears* in a word $w \in F(X)$ if w belongs to $F(X) \setminus F(X \setminus \{y\})$.

The identities in the following propositions follow from (2.2) and the definition of the Fox derivative. We will regularly use them without further notice.

Proposition 2.1 *For $s, t \in \overline{R}$, the following identities hold:*

- $\Psi\frac{\partial s^{-1}}{\partial x} \simeq -\Psi\frac{\partial s}{\partial x}$
- $\Psi\frac{\partial st}{\partial x} \simeq \Psi\frac{\partial s}{\partial x} + \Psi\frac{\partial t}{\partial x}$
- $\Psi\frac{\partial w^{-1}sw}{\partial x} \simeq \left(\Psi\frac{\partial s}{\partial x}\right)\psi(w), \quad w \in F(X)$

We now recall the definition of the Jacobian from [2]. For a finite presentation

$$\langle x_1, x_2, \dots x_g | f_1, f_2, \dots, f_m\rangle$$

of group G, we call the matrix

$$J = \begin{pmatrix} \Psi\frac{\partial f_1}{\partial x_1} & \Psi\frac{\partial f_1}{\partial x_2} & \dots & \Psi\frac{\partial f_1}{\partial x_g} \\ \Psi\frac{\partial f_2}{\partial x_1} & \Psi\frac{\partial f_2}{\partial x_2} & \dots & \Psi\frac{\partial f_2}{\partial x_g} \\ \vdots & \vdots & & \vdots \\ \Psi\frac{\partial f_m}{\partial x_1} & \Psi\frac{\partial f_m}{\partial x_2} & \dots & \Psi\frac{\partial f_m}{\partial x_g} \end{pmatrix}$$

the Jacobian of the presentation. In [2], Fox showed that Jacobians generated by two distinct presentations of the same group are *equivalent* matrices, where equivalence is as described below in Definition 2.2 (b). This is done by showing that Tietze transformations keep the matrices in the same equivalence class. Recall Tietze's criterion for a generator y to be removable from a presentation, (T4): the only relator the generator appears in, must have the form $y^{-1}w$ where $w \in F(X \setminus \{y\})$. Hence,

$$\Psi \frac{\partial y^{-1} w}{\partial y} = -1$$

and the derivative of the other relators are zero with respect to y.

The motivation behind our work is to construct a pathway in the opposite direction and possibly a one-to-one correspondence between presentations of a group and an equivalence class of matrices via Jacobians.

Definition 2.2

(a) *For positive integers* k *and* l, *by right linear combinations of*

$$a_1, a_2, \ldots, a_l \in \mathbb{Z}G^k,$$

we refer to terms

$$a_1 b_1 + a_2 b_2 + \ldots + a_l b_l \in \mathbb{Z}G^k$$

where $b_i \in \mathbb{Z}G$.

(b) *Given a Jacobian* J, *a matrix* J' *is said to be equivalent to* J *if a sequence of the following elementary operations can be applied to* J *to obtain* J'.

(I) *Permute rows or columns.*

(II) *Adjoin a new row to the matrix and the new row can be written as a right linear combination of other rows.*

(III) *Remove a row, a row that can be written as a right linear combination of the other rows of the matrix.*

(IV) *Adjoin a column and row to the matrix, where the intersection of the new row and column is* -1 *and the other entries of the column are zeros:*

$$M \longrightarrow \begin{pmatrix} M & 0 \\ * & -1 \end{pmatrix}.$$

(V) *Remove a column, a column that has a unique non-zero entry of* -1 *is removed along with the respective row*

$$\begin{pmatrix} M & 0 \\ * & -1 \end{pmatrix} \longrightarrow M.$$

Notice that operation (I) corresponds to permuting the generators or relators and does not change the group isomorphism type and thereby does not affect our discussion. Moreover, operations (II) and (III) correspond to the addition and removal of a relator by (T1) and (T2). However, the two remaining operations (IV) and (V) obstruct the desired correspondence. Operation (IV) does not necessarily correspond to adjoining a generator directly since the entries of the new row are a random set of elements in $\mathbb{Z}G$ and need not be derivatives of a word in $F(X)$. However, if they are derivatives of a word $w \in F(X)$, then the operation can be thought of adjoining a new generator y and its corresponding relator $y^{-1}w$.

The case which interests us is that of operation (V). The issue here is that simple examples can be found where the removal of a column of qualifying form, with a unique non-zero entry -1, and its corresponding row in the Jacobian can not be tracked back to the removal of a generator. The main problem being that the matrix is evaluated at $\mathbb{Z}G$ and if $\Psi\frac{\partial w}{\partial y} \simeq 0$, one can not conclude that the word w does not include powers of y. Furthermore, not every word f with $\Psi\frac{\partial f}{\partial y} \simeq -1$ has the form $y^{-1}w$.

Example 2.3 Consider the presentation

$$\left\langle a, b, y \mid a^3, b^2, abab, ay^{-1}ay^{-1}ay^2, a^3y^{-1}a^2bay^{-1}ay^{-1}ay^2b \right\rangle$$

for the dihedral group D_6. By consecutive use of the fourth, second and first relators, it's easy to derive from the fifth relator that

$$y^{-1}a^2 \simeq a^3(y^{-1}a^2(b(ay^{-1}ay^{-1}ay^2)b)) \in \overline{R}$$

Hence, using $y \simeq a^2$, we calculate the derivative of the relators accordingly;

$$\frac{\partial a^3}{\partial y} = \frac{\partial b^2}{\partial y} = \frac{\partial abab}{\partial y} = 0 \quad \Rightarrow \quad \Psi\frac{\partial a^3}{\partial y} \simeq \Psi\frac{\partial b^2}{\partial y} \simeq \Psi\frac{\partial abab}{\partial y} \simeq 0$$

$$\frac{\partial ay^{-1}ay^{-1}ay^2}{\partial y} = -y^{-1}ay^{-1}ay^2 - y^{-1}ay^2 + y + 1$$

$$\Psi\frac{\partial ay^{-1}ay^{-1}ay^2}{\partial y} \simeq -a^2 - 1 + a^2 + 1 \simeq 0$$

$$\frac{\partial a^3y^{-1}a^2bay^{-1}ay^{-1}ay^2b}{\partial y} = -y^{-1}a^2bay^{-1}ay^{-1}ay^2b - y^{-1}ay^{-1}ay^2b - y^{-1}ay^2b + yb + b$$

$$\Psi\frac{\partial a^3y^{-1}a^2bay^{-1}ay^{-1}ay^2b}{\partial y} \simeq -1 - a^2b - b + a^2b + b \simeq -1$$

First observe that $ay^{-1}ay^{-1}ay^2$ has four appearances of y but its derivative is 0. Additionally, $a^3y^{-1}a^2bay^{-1}ay^{-1}ay^2b$ does not have the form $y^{-1}w$