REIΦυσις

MODELLI TEORICI E COMPUTAZIONALI DI FISICA DELLA MATERIA

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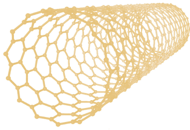

Conoscere il comportamento di atomi e molecole permette la comprensione profonda della relazione fra materia, forza ed energia.

Roberto Zivieri

Negli ultimi decenni il settore scientifico della Fisica della materia è stato attraversato da una febbrile attività di ricerca, rivolta alla comprensione dei fenomeni fondamentali.

Molti dei risultati ottenuti, sia mediante formulazione di modelli teorici e computazionali, sia attraverso esperimenti, sono ancora in fase di conferma e di verifica. La descrizione dettagliata del formalismo matematico che sta alla base dei modelli formulati richiede ampi spazi di approfondimenti. Lo scopo principale della collana ReiΦυσις (dal latino *rei*, dativo di [res, rei], "cosa, materia" e dal greco φυσις, [physis], "fisica, natura") è, pertanto, quello di raccogliere queste conoscenze e diffonderle ai lettori specialisti del settore, fornendo una panoramica il più possibile esauriente sulla stato dell'arte.

La collana accoglie studi, ricerche e scoperte rilevanti della Fisica della materia e trasmettere input e stimoli per l'ideazione e lo sviluppo di ricerche future, sia di settore che in campi affini o con esso interrelati. In ReiΦυσις saranno accolti anche volumi e monografie di carattere meno settoriale e specialistico, che risultino introduttivi per il lettore non specialista, dotato di una conoscenza base di Fisica generale propedeutica alla comprensione di tematiche più specialistiche di Fisica teorica della materia.

Ampio spazio è dedicato alle potenziali applicazioni tecnologiche che originano da questa branca della fisica, come le memorie magnetiche e i computer quantistici: non un mero completamento delle conoscenze scientifiche, ma esempi di sviluppo e progresso tecnologico che confluiscono e contribuiscono all'evoluzione della società stessa.

Web content

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Laser–Plasma Acceleration and Secondary EM Radiation

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Contents

Introduction

The technological advent of high-intensity lasers has paved the way to the experimental investigation of relativistic optics. With the name of relativistic optics we refer to the interaction of matter, in particular of electrons, with powerful pulses of light as short as few tens of femtoseconds, able to transfer relativistic momenta to the charges. A great variety of non-linear phenomena unknown to classical optics has been discovered and explored in last decades, generating new hybrid sciences like the one which joins together the physics of laser-matter interaction with the accelerator physics. This new science inherits from the accelerator physics the main goal to produce high-field gradients, in order to boost-up, in the most clever and economic way, the energy and the quality of accelerated particle bunches. Nevertheless it shows the novelty to do it in very unconventional manners, and within compact space-scales. It 's worth specifying that, to date, the state of the art of these new concepts of accelerator physics are promising but not yet competitive with standard techniques and technologies up to the point to be already considered a valid, robust alternative. The laser-matter interaction at relativistic intensities is furthermore a discipline which necessary pairs with another fundamental branch of physics, which is the plasma physics. This is due to the fact that the electric fields associated to high-intensity ($\gtrsim 10^{18}$ W/cm^2) lasers are comparable to and often greater than the atomic fields which keep bound the atomic electrons. This implies that ionization of matter and plasma formation are phenomena normally observed during laser-matter interactions in this regime. Actually, plasmas offer a natural hosting environment for electric gradients even many orders larger than the atomic fields, overcoming in this sense the limitations by which standard accelerating Radio Frequency (RF) cavities are affected.

Chapter 1

Introductory plasma physics for acceleration

The states of the matter are usually addressed only as "solid", "liquid" and "gaseous", the last two often undergoing the unique appellation "fluid". The solid state differs from the fluid state for the correlation range among its constituents: the solid state of some element is usually "more ordered" than the fluid state of the same specimen. In principle if with "matter" we mean a system of neutral atoms, arbitrarily correlated/ordered, even in excited states, then plasmas cannot be considered rigorously a state of the matter, but some kind of ionized matter. Actually plasmas are sometimes addressed as the fourth state of the matter. Whatever is the right way to define it, a plasma is ionized matter which preserves its global neutrality, a state in which the matter is not anymore locally constituted by neutral atoms but by a mixture of ionized atoms and electrons (in the extreme case by bare nuclei and electrons), in some circumstances acting like a collectivity. Plasmas are interesting for the acceleration of charged particles in terms of the accelerating gradients they can sustain. We will show later on that plasmas can sustain electric fields, under the form of collective electrostatic waves, even greater than the value E_{wb}, called "cold nonrelativistic wavebreaking field", defined as:

$$E_{wb} = \frac{m_e \omega_p c}{e} \sim 96\sqrt{n_e[cm^{-3}]}[V/m] \tag{1.1}$$

where m_e, e and c are the electron mass, the elementary charge and the velocity of light in vacuum respectively, n_e is the electron plasma density, while the electron plasma frequency is defined as:

$$\omega_p^2 = \frac{n_e e^2}{m_e \varepsilon_0} \tag{1.2}$$

where ε_0 is the vacuum dielectric constant. Typical electron plasma densities for acceleration experiments range between $10^{15} \div 10^{23} cm^{-3}$: by inspection of 1.1 we can notice that, in principle, accelerating gradients can be reached in plasmas many order of magnitude grater than in conventional RF cavities.

1.1 The Boltzmann-Vlasov equation

Since we are especially interested in plasmas which show collective behaviors we can approach a macroscopic statistical description. We start by considering the classical (nonrelativistic) single particle phase space density $f(\vec{r}, \vec{v}, t)$, which can be referred to an electron as well as to an ion. We introduced the position and velocity vectors $\vec{r}$ and $\vec{v}$ at the time t. The Liouville theorem states that f is a time constant when collisions are negligible. When the electron-ion collisions are taken into account the Liouville theorem takes the form of the Boltzmann equation:

$$\dot{f} = \frac{df}{dt} = \left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_r + \dot{\vec{v}} \cdot \vec{\nabla}_v\right] f = \left(\frac{\partial f}{\partial t}\right)_c \tag{1.3}$$

where $\vec{\nabla}_{r,v}$ are the gradients with respect the position and velocity vectors respectively, while the right term of the Eq. 1.3 represents the rate at which the phase space is deformed due to the electron-ion collisions. In the opposite case when the collisions are not important we get the Vlasov equation:

$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_r + \dot{\vec{v}} \cdot \vec{\nabla}_v\right] f = 0 \tag{1.4}$$

The method to derive macroscopic variables from the microscopic phase space density f is to take velocity moments. In this section we derive macroscopic equations by taking velocity moments of the Boltzmann equation. The collision term is retained in the analysis in order to give a more complete macroscopic theory. When not needed, the collision term is readily neglected. We first calculate the phase space density moments, observing that:

$$n = \int d^3v \; f \tag{1.5}$$

is nothing but the particle density, then

$$n\vec{u} = \int d^3v \; \vec{v} f \tag{1.6}$$

is the mean density flux, and finally

$$\hat{P}_{ij} = m \int d^3v \; (v_i - u_i)(v_j - u_j) f \tag{1.7}$$

is the plasma pressure tensor, where $i, j = x, y, z$ indicate the vector components and m is the particle mass. Now we take the moments of the Boltzmann equation 1.3 with respect to the velocity $\vec{v}$. The zeroth momentum gives the continuity equation:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0 \tag{1.8}$$

The zeroth order momentum of the collision term in 1.3 has been considered zero by neglecting ionizing or recombining collisions otherwise Eq. 1.8 should take the following form:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = n(\nu_i - \nu_r) \tag{1.9}$$

where $\nu_{i,r}$ are the ionization and recombination rate respectively. In laser-fusion plasmas could be also necessary to consider the fusion rate in Eq. 1.9. The first order momentum of the Boltzmann equation gives the force equation:

$$nm\frac{\partial \vec{u}}{\partial t} + nm(\vec{u} \cdot \vec{\nabla})\vec{u} = n\vec{F}_{ext} - \vec{\nabla}P + m\int d^3v\ \vec{v}\Big(\frac{\partial f}{\partial t}\Big)_c \tag{1.10}$$

where we assumed for simplicity an isotropic pressure ($\hat{P}_{ij} = P\delta_{ij}$ with δ_{ij} the Kronecker delta) and an external force $\vec{F}_{ext}$ acting on the system. The collision term in Eq. 1.10 can be evaluated in the Bhatnagar-Gross-Krook (BGK [1]) approximation:

$$\int d^3v\ \vec{v}\Big(\frac{\partial f}{\partial t}\Big)_c \sim n\sum_j \nu_j(\vec{u} - \vec{u}_j) \tag{1.11}$$

where ν_j and $\vec{u}_j$ are the collision rate with the j-species and the mean velocity of the j-species respectively. We notice that every n^{th} moment of the Boltzmann equation makes to appear the $(n+1)^{th}$ moment of the phase space density in the resulting macroscopic equation. Therefore, in principle, we could continue with this procedure to infinity but for practical reasons we need to truncate our game at some point. This is possible by introducing the concept of equation of state. The second moment of the Boltzmann equation would give an equation relating the pressure evolution to the heat flow 3^{rd} rank tensor. Once one has an equation of state, the pressure tensor is known and defined, so the equation for the heat flow is not necessary. The equation of state together with the equations 1.8, 1.10 for each species in the plasma (the equations are coupled when the collisions are taken into account) and the external field equations , like Maxwell equations in the case of electromagnetic interaction, form a complete set of equations for the description of the plasma system. On the other hand the heat flow cannot be neglected when the heat transport and the diffusion are important. We come back later at this point because for most of the applications of our interest we will consider relativistic cold plasmas where the pressure can be neglected. Furthermore the equation of state is not valid when the Local Thermodynamic Equilibrium (LTE) approximation cannot be considered, i.e. when the collision rate are smaller than the radiative rate and in general than the rate of any dissipative process. In that case other solutions have to be find for the truncation. Since we are interested in relativistic plasmas, we need to generalize Eq. 1.10 to the relativistic case. In order to do this we define the energy-momentum tensor for the relativistic fluid:

$$T^{\mu\nu} = (\epsilon c^2 + P)u^\mu u^\nu + Pg^{\mu\nu} \tag{1.12}$$

where the relativistic mass density has been introduced $\epsilon = n\gamma m$, the four-velocity $u^\mu = (\gamma, \gamma\vec{\beta})$ with $\gamma = (1-\beta^2)^{-1/2}$ the Lorentz factor and $\vec{\beta} = \vec{u}/c$ the normalized velocity ($\vec{u}$ is the mean velocity of the fluid like before, to be not confused with the spatial component of the four-vector u^μ). The indexes μ, ν run from 0 to 3, 0 for the temporal component and $1, 2, 3$ for x, y, z components respectively. The Minkowski tensor $g^{\mu\nu}$ is considered for a flat spacetime (we are in the case where gravitational forces are completely negligible). The conservation of energy and momentum is expressed as:

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.13}$$

In order to obtain an equation for the relativistic momentum we first define the tensor:

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{1.14}$$

Then we consider the equation:

$$h_{\alpha\nu}\partial_\mu T^{\mu\nu} = 0 \tag{1.15}$$

where $\alpha = 1, 2, 3$. This equation can be recasted in a more explicit form, by also adding correctly the interaction with an external force. The external four-force per unit volume is $\epsilon c^2 w^\mu$ where $w^\mu = du^\mu/ds = d^2x^\mu/ds^2$ is the four-acceleration and $ds^2 = c^2t^2 - x^2 - y^2 - z^2$. Therefore the four-force per unit volume can be expressed as $n\gamma^2 mcdu^\mu/dt = n\gamma^2 dp^\mu/dt$ and the spatial component is $n\gamma^2\vec{F}_{ext}$. The Eq. 1.15 becomes:

$$(\epsilon c^2 + P)u^\mu\partial_\mu\gamma\vec{\beta} = -\vec{\nabla}P - \gamma\vec{\beta}u^\mu\partial_\mu P + n\gamma^2\vec{F}_{ext} \tag{1.16}$$

Let's neglect, for the same reasons above, the pressure terms:

$$\epsilon c^2 u^\mu\partial_\mu\vec{u} = n\gamma^2 mc\Big[\partial_t + c\vec{\beta}\cdot\vec{\nabla}\Big]\vec{u} = n\gamma^2\vec{F}_{ext} \tag{1.17}$$

which finally, once adding the collision term, gives:

$$n\frac{d\vec{p}}{dt} = n\vec{F}_{ext} + n\sum_j \nu_j(\vec{p} - \vec{p}_j) \tag{1.18}$$

where the relativistic momentum is $\vec{p} = \gamma m\vec{\beta}c$. The equation 1.18 is the relativistic generalization of equation 1.10. We will make use of this to derive the interaction between strong electromagnetic fields, driven by high intensity lasers, and plasmas, in particular the generation of electron plasma waves.

1.2 The Debye length

Among the various classes into which plasmas can be subdivided, one of the most important is related to the electron-ion collisions. Plasmas can be said either "non-collisional" in the sense that the electron-ion collision rate ν_{ei} is much smaller then the electron plasma frequency ω_p or "collisional" with the opposite meaning. The difference between the two kinds of plasmas resides in the motion of the particles, in particular the single electron trajectories are dominated by the action of local electric fields in the proximity of the ion centers in the case of collisional plasmas, while the electron move as a collectivity under the action of mean fields in the case of non-collisional plasmas. To be more quantitative the scale of the local electric fields in plasmas is given by the Debye length, usually expressed as:

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{n_e e^2}} \tag{1.19}$$

where k_B is the Boltzmann constant and T_e is the plasma electron temperature. Let's consider the Poisson equation for a plasma made by electrons and ions of different species with different degrees of ionization. Let's also consider a local displacement of the charges, induced by an external source of potential S_{ext}.

$$\nabla^2 \phi = \frac{e}{\varepsilon_0}(n_e - \sum_{ij} j n_{ij} - n_0) - S_{ext} \tag{1.20}$$

The term $n_0 = n_{0e} + \sum_{ij} j n_{0ij}$ is the initial unperturbed density, where n_{0e} and n_{0ij} are the initial unperturbed electron plasma density and the initial unperturbed plasma density of the ion species i carrying a net charge je. At the equilibrium, the force equations for the electron and the ion fluids give:

$$e n_e \vec{\nabla} \phi = -\vec{\nabla} P_e \tag{1.21}$$

$$j e n_{ij} \vec{\nabla} \phi = \vec{\nabla} P_{ij} \tag{1.22}$$

which combined with the equations of state for a perfect fluid at constant temperature

$$P_e = n_e k_B T_e \tag{1.23}$$

$$P_{ij} = n_{ij} k_B T_i \tag{1.24}$$

where $P_{e,ij}$ stays for the electron/ion pressure, finally give:

$$e n_e \vec{\nabla} \phi = -k_B T_e \vec{\nabla} n_e \tag{1.25}$$

$$j e n_{ij} \vec{\nabla} \phi = k_B T_i \vec{\nabla} n_{ij} \tag{1.26}$$

The solutions of the equations 1.25,1.26 for $je\phi << k_B T_{e,i}$ are the following:

$$n_e = n_{0e} \exp\Big(- \frac{e\phi}{k_B T_e}\Big) \sim n_{0e}\Big(1 - \frac{e\phi}{k_B T_e}\Big) \tag{2.27}$$

$$n_{ij} = n_{0ij} \exp\Big(\frac{je\phi}{k_B T_i}\Big) \sim n_{0ij}\Big(1 + \frac{je\phi}{k_B T_i}\Big) \tag{2.28}$$

By inserting the equations 1.27 and 1.28 in 1.20, we get:

$$\nabla^2\phi - \frac{\phi}{\lambda_D^2} = -S_{ext} \tag{1.29}$$

where the most general definition of Debye length is

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B}{e^2\left(\frac{n_{0e}}{T_e} + \sum_{ij} j^2 \frac{n_{0ij}}{T_i}\right)}} \tag{1.30}$$

Expression 1.30 reduces to 1.19 when the electrical mobility of the ions is neglected with respect to that of the electrons. The ion electrical mobility μ_i is defined as

$$\mu_{i,\alpha} = \frac{u_{i,\alpha}}{\partial_\alpha \phi} = \sqrt{\frac{T_i m_e}{T_e M_i}}\frac{u_{e,\alpha}}{\partial_\alpha \phi} = \sqrt{\frac{T_i m_e}{T_e M_i}}\mu_{e,\alpha} \tag{1.31}$$

where $\alpha = x, y, z$, M_i is the ion mass under consideration and μ_e is the electron electrical mobility. Eq. 1.31 supposes, for sake of simplicity, an ion with unitary charge. In fact if the ion density can be considered unperturbed on the time scale of the electron density perturbation, then $n_{ij} = n_{0ij}$ in 1.20. In other words, from the continuity equation (this holds both for electrons and ions), is possible to set a linear dependence between the mobility and the density displacement. In particular the density displacement, say Δn ($n_e - n_{e0}$ for electrons and $\sum_{ij} j(n_{ij} - n_{0ij})$ for ions, as shown above), can be expressed, for small external perturbations, as $\Delta n = n_0 \mu S_{ext}$, where n_0 in this case is the unperturbed density of the species under consideration and μ the respective mobility along the direction under study. The equation 1.31 holds at the thermal equilibrium. By Fourier-transforming 1.29 we obtain:

$$\widetilde{\phi}(\vec{k}) = \frac{\widetilde{S}_{ext}(\vec{k})}{\left(k^2 + \frac{1}{\lambda_D^2}\right)} \tag{1.32}$$

By taking the inverse Fourier transform of 1.32 we finally get the convolution integral:

$$\phi(\vec{r}) = \frac{1}{4\pi}\int_0^\infty d^3r' \frac{e^{-\frac{|\vec{r}-\vec{r}'|}{\lambda_D}}}{|\vec{r}-\vec{r}'|} S_{ext}(\vec{r'}) \tag{1.33}$$

For example, let's consider a point charge Ze potential source (with Z integer) positioned at the origin of our reference system well inside the plasma volume, i.e.:

$$S_{ext} = \frac{Ze}{\varepsilon_0}\delta(\vec{r}) \tag{1.34}$$

In this case, by inserting 1.34 in 1.33 we obtain the well-known formula of the screened potential of an ion charge in a plasma medium:

$$\phi = \frac{Ze}{4\pi\varepsilon_0 r} e^{-\frac{r}{\lambda_D}} \tag{1.35}$$

Therefore by having a look to Eq. 1.33, we can state that, at the thermal equilibrium, each local electric field present in a plasma can be effective only on a scale defined by the Debye length. This is due to a screening mechanism provided by the electrons. The more numerous are the electrons inside a sphere of radius λ_D, the more efficient will be the electrical screening and the electrons will move like a collectivity with a resonance frequency given by ω_p, being the electron-ion collisions more unlikely. The so called plasma parameter N_D

$$N_D = \frac{4\pi}{3} n_e \lambda_D^3 \sim 1.7 \times 10^9 \sqrt{\frac{T_e^3[eV]}{n_e[cm^{-3}]}} \tag{1.36}$$

takes into account for this, determining whether a plasma acts like a collectivity ($N_D >> 1$) or not ($N_D \lesssim 1$).

1.3 Electron plasma waves and electromagnetic waves

In this section we consider the plasma interaction with an external electromagnetic field, for example driven by a high intensity laser. We are going to consider the ion fluid motionless, inert, only like a neutralizing background, because we are interested in very fast phenomena which don't involve the ion dynamics. Furthermore the electrons are considered cold, in the sense that the kinetic energy acquired via the interaction with the electromagnetic field is much greater than their initial thermal energy: in this case we are authorized to neglect the pressure term in the equation 1.10. Finally we consider plasmas where the electron plasma frequency ω_p is much greater than the electron-ion collision frequency ν_{ei}, so that collisions are not important during the electron fluid collective oscillations. The last assumption also corresponds to the condition $N_D >> 1$, derived in the previous section. Plasmas we study in this section are even not magnetized, so that the magnetic permeability is that of vacuum μ_0. The complete system of equations we need to describe the generation of electron plasma waves during the interaction of the plasma with an electromagnetic field is:

$$\frac{d\vec{p}}{dt} = -e(\vec{E} + \vec{u} \times \vec{B}) \tag{1.37}$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}) = 0 \tag{1.38}$$

$$\vec{\nabla} \cdot \vec{E} = -e\frac{n_e - n_0}{\varepsilon_0} \tag{1.39}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.40}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.41}$$

$$\vec{\nabla} \times \vec{B} = -\mu_0 e n_e \vec{u} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{1.42}$$

where n_e is the electron density, n_0 the initial unperturbed electron density, ε_0 the vacuum dielectric constant, $\vec{E}$ and $\vec{B}$ the electric and magnetic field respectively. We impose that all the quantities depend on the only variable $\zeta = z - v_\phi t$, where v_ϕ is the phase velocity of the electromagnetic field inside the plasma. The equations become:

$$(u_z - v_\phi)\partial_\zeta \vec{p} = -e(\vec{E} + \vec{u} \times \vec{B}) \tag{1.43}$$

$$\partial_\zeta \Big[(u_z - v_\phi)n_e\Big] = 0 \tag{1.44}$$

$$\partial_\zeta E_z = -e\frac{n_e - n_0}{\varepsilon_0} \tag{1.45}$$

$$\partial_\zeta B_z = 0 \tag{1.46}$$

$$\vec{\nabla} \times \vec{E} = v_\phi \partial_\zeta \vec{B} \tag{1.47}$$

$$\vec{\nabla} \times \vec{B} = -\mu_0 e n_e \vec{u} + \beta_\phi^2 \partial_\zeta \vec{E} \tag{1.48}$$

where $\beta_\phi = v_\phi/c$. The z component of Eq. 1.48 is zero, so by combining with Eq. 1.45 we obtain:

$$n_e = \frac{v_\phi n_0}{v_\phi - u_z} \tag{1.49}$$

From Eq. 1.47 we get the identities $E_y = -v_\phi B_x$ and $E_x = v_\phi B_y$: by considering in virtue of these identities the transverse components of Eq. 1.43 and also noting that $B_z = 0$ when no external static magnetic fields are acting on the system, we obtain:

$$B_y = \frac{1}{e}\partial_\zeta p_x \tag{1.50}$$

and

$$B_x = -\frac{1}{e}\partial_\zeta p_y \tag{1.51}$$

Now by considering the transverse components of the Eq. 1.48 and rearranging we get:

$$\partial_\zeta B_x = -\frac{\mu_0 e n_e u_y}{1-\beta_\phi^2} \tag{1.52}$$

and

$$\partial_\zeta B_y = \frac{\mu_0 e n_e u_x}{1-\beta_\phi^2} \tag{1.53}$$

By combining the equations 1.50, 1.51, 1.52, 1.53 is possible to obtain an equation for the transverse moments of the electron fluid:

$$\partial_\zeta^2 p_{x,y} = \gamma_\phi^2 m k_p^2(\zeta) u_{x,y} \tag{1.54}$$

where $k_p = \omega_p/c$ is the electron plasma wave number and $\gamma_\phi = (1-\beta_\phi^2)^{-1/2}$ is the Lorentz factor associated to the electromagnetic wave. The equation for the longitudinal momentum is found by differentiating Eq. 1.43 and substituting $B_{x,y}$ from equations 1.50,1.51 and n_e from Eq. 1.49:

$$\partial_\zeta\Big[(u_z - v_\phi)\partial_\zeta p_z\Big] = \frac{e^2}{\varepsilon_0}\frac{n_0 u_z}{v_\phi - u_z} - u_x\partial_\zeta^2 p_x - \partial_\zeta p_x \partial_\zeta u_x - u_y \partial_\zeta^2 p_y - \partial_\zeta p_y \partial_\zeta u_y \tag{1.55}$$

The equations 1.54 and 1.55 describe completely the dynamics of the electron plasma oscillations for an arbitrary intensity of the driving electromagnetic field. Nevertheless they are not analytically solvable, except in particular cases, for example in the case of small density perturbation (corresponding to small intensity electromagnetic fields driving the perturbation) and in the 1D case for an arbitrary pump strength as we see in next subsections. Once the the momentum is known the electric and magnetic fields can calculated as:

$$E_x = \frac{v_\phi}{e}\partial_\zeta p_x \tag{1.56}$$

$$E_y = \frac{v_\phi}{e}\partial_\zeta p_y \tag{1.57}$$

$$E_z = -\frac{e n_0}{\varepsilon_0}\int_{-\infty}^{\zeta} d\zeta' \frac{u_z(\zeta')}{v_\phi - u_z(\zeta')} \tag{1.58}$$

and the non-zero components of the magnetic field are given by the equations 1.50 and 1.51.

1.3.1 Linear plasma waves and propagation of electromagnetic waves

In order to explore the nature of the plasma electron collective oscillations we first consider the simple case of interaction with a linearly polarized electromagnetic wave, for example in the x direction, in the limits $n_e \sim n_0 + \delta n_e$, $u_x, z << v_\phi$, $p_y, u_y = 0$, $u_{x,z} \sim p_{x,z}/m_e$, $u_{x,z} \sim u_{x0,z0} + \delta u_{x,z}$, $p_{x,z} \sim p_{x0,z0} + \delta p_{x,z}$, and $v_\phi \sim c$, where the quantities with the subscript 0 refer to the unperturbed situation and therefore they are constant. The Eq. 1.55 can be written in this approximation:

$$\partial_\zeta^2 \delta u_z = -k_{p0}^2 \delta u_z - \frac{1}{2c} \partial_\zeta^2 \delta u_x^2 \tag{1.59}$$

By recalling that the transverse canonical momentum is invariant during the interaction between a plasma electron and an electromagnetic wave, we set $u_x = eE_x/m_e\omega_0 = ac$, which is valid when the electron has zero momentum before the interaction. We defined ω_0, the pulsation of the electromagnetic wave and the normalized nonrelativistic quivering velocity $a = eE_x/m_e\omega_0 c$ of the electron. Therefore the Eq. 1.59 becomes:

$$\partial_\zeta^2 \delta u_z = -k_{p0}^2 \delta u_z - \frac{c}{2} \partial_\zeta^2 a^2 \tag{1.60}$$

The Eq. 1.60 is resolved by the Green function method. The Green function $G(\zeta - \zeta')$ for the Eq. 1.60 is defined as:

$$\partial_\zeta^2 G + k_{p0}^2 G = \delta(\zeta - \zeta') \tag{1.61}$$

By Fourier transforming the Eq. 1.61 we obtain:

$$G(k) = -\frac{1}{k^2 - k_{p0}^2} \tag{1.62}$$

from which

$$G(\zeta - \zeta') = \frac{1}{k_{p0}} \sin[k_{p0}(\zeta - \zeta')] sign(\zeta - \zeta') \tag{1.63}$$

where the sign function $sign(\zeta - \zeta')$ is equal to one because ζ' is always chosen behind the electromagnetic wave positioned at ζ, so that $\zeta - \zeta' > 0 \; \forall \; \zeta, \zeta'$. The solution of the Eq. 1.60 is finally found:

$$\delta u_z = \frac{c}{2k_{p0}} \int_{-\infty}^{\zeta} d\zeta' \sin[k_{p0}(\zeta - \zeta')] \partial_{\zeta'}^2 a^2(\zeta') \tag{1.64}$$

The corresponding electric field is calculated via the Eq. 1.58.

$$E_z = -\frac{en_0}{k_{p0}\varepsilon_0} \int_{-\infty}^{\zeta} d\zeta' \int_{-\infty}^{\zeta'} d\zeta'' \sin[k_{p0}(\zeta' - \zeta'')] \partial_{\zeta''}^2 \frac{a^2}{2}(\zeta'') \tag{1.65}$$

The coefficient in front of the integral in the Eq. 1.65 can be recasted as $en_0/k_{p0}\varepsilon_0 = m_e\omega_p c/e = E_{wb}$, showing that the Eq. 1.1 sets the limit value of the amplitude of the longitudinal electric fields in the linear regime of the electron plasma oscillations. The Eq. 1.64 shows that an electromagnetic wave