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From Groups to Leibniz Algebras: Common Approaches, Parallel Results

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Abstract

In this article, we study (locally) nilpotent and hyper-central Leibniz algebras. We obtained results similar to those in group theory. For instance, we proved a result analogous to the Hirsch-Plotkin Theorem for locally nilpotent groups.

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1 Introduction

The concept of nilpotency arises in many algebraic disciplines and plays a key role there. One of the sources of its origin were triangular matrices. The ring theoretical concept of a commutator of two triangular matrices led to the zero-triangular matrices, the nilpotency in associative rings, the lower central series, and the concept of nilpotency in Lie algebras. The concept of a group-theoretical commutator of two nonsingular triangular matrices led to unitriangular matrices, and to the concept of the lower central series in the group of matrices. At the first stage, this commonality of origin brought some parallelism in approaches, however then the specificity of each theory introduces its own modifications. Nevertheless, it turned out that in many cases, the same approaches led to comparable results in groups and Lie algebras. This parallelism runs through the book [1], it was noted in many articles devoted to Lie algebras, in particular, in the paper [17]. One of the interesting generalizations of Lie algebras is Leibniz algebras. Therefore, the following question naturally arises: Which of the group-theoretical concepts and results have analogs in Leibniz algebras? An algebra L over a field F is said to be a *Leibniz algebra* (more precisely a *left Leibniz algebra*) if it satisfies the Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]] \qquad \text{for all } \ a, b, c \in L. \quad \text{(LI)}$$

Leibniz algebras are generalizations of Lie algebras. Indeed, a Leibniz algebra L is a Lie algebra if and only if $[a, a] = 0$ for every element $a \in L$. By this reason, we may consider Leibniz algebras as "non-anticommutative" analogs of Lie algebras. Leibniz algebras appeared first in the papers of A.M. Bloh [4],[5],[6],... in which he called them the D-algebras. However, at that time these researches were not in demand, and they have not been properly developed. Real interest in Leibniz algebras arose only after two decades. This happened thanks to J.L. Loday [12], who "rediscovered" these algebras and used the term *Leibniz algebras* since it was Leibniz who discovered and proved the "Leibniz rule" for differentiation of functions.

The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. The theory of Leibniz algebras develops quite intensively now, however, it should be noted that most of the obtained results refer to finite-dimensional Leibniz algebras, and in the greater part of the latter, algebras over fields of characteristic zero are only considered. This also applies to nilpotent Leibniz algebras. The concept of nilpotency for the Leibniz algebras is introduced as follows. Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A, B]$ will denote a subspace, generated by all elements $[a, b]$ where $a \in A, b \in B$. We note that if A is an ideal of L, then $[A, A]$ is also an ideal of L.

If M is non-empty subset of L, then $\langle M \rangle$ denotes the subalgebra of L generated by M.

Let L be a Leibniz algebra. We define the lower central series of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \geqslant \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L, \gamma_2(L) = [L, L]$, and recursively,

$$\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$$

for all ordinals α, while

$$\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$$

for limit ordinals λ. It is possible to shows that every term of this series is an ideal of L. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots, L] \ldots]$.

A Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote by $ncl(L)$ the nilpotency class of L.

In some algebraic structures, another definition of nilpotency based on the concept of the (upper) central series is used. In fact, suppose that L is a nilpotent Leibniz algebra and $\gamma_{k+1}(L) = \langle 0 \rangle$. For each factor $\gamma_j(L)/\gamma_{j+1}(L)$ we have

$$[L, \gamma_j(L)] = \gamma_{j+1}(L) \quad \text{and} \quad [\gamma_j(L), L] \leqslant \gamma_{j+1}(L),$$

and this leads us to the following concepts. Let A, B be the ideal of L such that $A \leqslant B$. The factor B/A is called *central* (in L) if

$$[L, B], [B, L] \leqslant A.$$

The center $\zeta(L)$ of a Leibniz algebra L is defined in the following way:

$$\zeta(L) = \{x \in L \,|\, [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

Clearly, $\zeta(L)$ is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$. Starting from the center we can define the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \ldots \leqslant \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively

$$\zeta_{\alpha+1}(L)/\zeta_{\alpha}(L) = \zeta(L/\zeta_{\alpha}(L))$$

for all ordinals α, while

$$\zeta_{\lambda}(L) = \bigcup_{\mu<\lambda} \zeta_{\mu}(L)$$

for limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_{\infty}(L)$ of this series is called the *upper hypercenter* of L. A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter. Denote by $zl(L)$ the length of upper central series of L. In the paper [11], the connection between the lower and upper central series in nilpotent Leibniz algebras has been considered. It was proved that in this case, the lengths of the lower and upper central series coincide. Moreover, they are the least among the lengths of all other central series.

The concepts of upper and lower central series introduced here immediately lead to the following classes of Leibniz algebras.

A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

A Leibniz algebra L is said to be *hypocentral* if it coincides with the lower hypercenter.

In the case of finite dimensional algebras, these two concepts coincide, but in general, these two classes are very different. Thus, for finitely generated hypercentral Leibniz algebras we have the following theorem.

Theorem A. *Let* L *be a finitely generated Leibniz algebra over a field* F*. If* L *is hypercentral, then* L *is nilpotent. Moreover,* L *has finite dimension. In particular, a finitely generated nilpotent Leibniz algebra has finite dimension.*

This result is an analog of a similar group theoretical result proved by A.I. Mal'cev (see [13]).

At the same time, a finitely generated hypocentral Leibniz algebra can have infinite dimension. Thus, a cyclic Leibniz algebra $\langle a\rangle$ where an element a has infinite depth is hypocentral and has infinite dimension (see [8]).

A Leibniz algebra L is said to be *locally nilpotent* if every finite subset of L generates a nilpotent subalgebra.

That is why, hypercentral Leibniz algebras give us examples of locally nilpotent algebras. We obtained the following characterization of hypercentral Leibniz algebras.

Theorem B. *Let* L *be a Leibniz algebra over a field* F. *Then* L *is hypercentral if and only if for each element* $a \in L$ *and every countable subset* $\{x_n | n \in \mathbb{N}\}$ *of elements of* L *there exists a positive integer* k *such that all commutators* $[x_1, \dots, x_j, a, x_{j+1}, \dots, x_k]$ *are zeros for all* $j, 0 \leqslant j \leqslant k$.

Corollary. *Let* L *be a Leibniz algebra over a field* F. *Then* L *is hypercentral if and only if every subalgebra of* L *having finite or countable dimension is hypercentral.*

These results are analogues to the results proved for groups by S.N. Chernikov (see [7]).

Let L be a Leibniz algebra. If A, B are nilpotent ideals of L, then their sum $A + B$ is a nilpotent ideal of L (see [3], Lemma 1.5). In this connection, the following question arises: is an analogous assertion valid for locally nilpotent ideals? As it was shown by B. Hartley (see [9]), for Lie algebras this assertion takes place. Our next result gives a positive answer to this question.

Theorem C. *Let* L *be a Leibniz algebra over a field* F, A, B *be locally nilpotent ideals of* L. *Then* $A + B$ *is locally nilpotent.*

Corollary C1. *Let* L *be a Leibniz algebra over a field* F *and* S *be a family of locally nilpotent ideals of* L. *Then a subalgebra generated by* S *is locally nilpotent.*

Corollary C2. *Let* L *be a Leibniz algebra over a field* F. *Then* L *has the greatest locally nilpotent ideal.*

Let L be a Leibniz algebra over field F. The greatest locally nilpotent ideal of L is called the *locally nilpotent radical* of L and will be denoted by Ln(L).

These results are the analogues to the results for groups proved by K.A. Hirsch (see [10]) and B.I. Plotkin (see [15]); see also the survey [16].

The subalgebra Nil(L) generated by all nilpotent ideals of L is called the *nil-radical* of L. Clearly Nil(L) is an ideal of L. If L = Nil(L), then L is called a Leibniz *nil-algebra*. Every nilpotent Leibniz algebra is a nil-algebra, but the converse is not true even for a Lie algebra. Every Leibniz nil-algebra is locally nilpotent, but converse is not true even for a Lie algebra. Moreover, there exists a Lie nil-algebra, which is not hypercentral (see, for example, [1], Chapter 6).

Note the following important properties of locally nilpotent Leibniz algebras.

Theorem D. *Let* L *be a locally nilpotent Leibniz algebra over a field* F*.*

(i) *If* A, B *are ideals of* L *such that* $A \leqslant B$ *and the factor* B/A *is* L*-chief, then* B/A *is central in* L *(that is* $B/A \leqslant \zeta(L/A)$*). In particular, we have that* $\dim_F(B/A) = 1$.

(ii) *If* A *is a maximal subalgebra of* L*, then* A *is an ideal of* L*.*

Let L be a Leibniz algebra over a field F and H a subalgebra of L. The *idealizer* of H is defined by the following rule:

$$\mathbb{I}_L(H) = \{x \in L \mid [h, x], [x, h] \in H \text{ for all } h \in H\}.$$

It is possible to prove that the idealizer of H is a subalgebra of L. If L is a hypercentral (in particular, nilpotent) Leibniz algebra, then $H \neq \mathbb{I}_L(H)$ (see Proposition 1.10 below). This leads us to the following class of Leibniz algebras.

Let L be a Leibniz algebra over field F. We say that L *satisfies the idealizer condition* if $\mathbb{I}_L(A) \neq A$ for every proper subalgebra A of L.

A subalgebra A is called *ascendant* in L, if there is an ascending chain of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \dots A_\alpha \leqslant A_{\alpha+1} \leqslant \dots A_\gamma = L$$

such that A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$.

It is possible to prove that L satisfies the idealizer condition if and only if every subalgebra of L is ascendant. The last our result is the following

Theorem E. *Let* L *be a Leibniz algebra over a field* F*. If* L *satisfies the idealizer condition then* L *is locally nilpotent.*

This result is an analogue to the result proved for groups in [14] by B.I. Plotkin.

Again, it should be noted that Leibniz algebras with the idealizer condition will form a subclass of the class of locally nilpotent Leibniz algebras, since this is already the case for Lie algebras (see, for example, [1], Chapter 6).

2 On hypercentral Leibniz algebras

Proposition 2.1 *Let* L *be a finitely generated Leibniz algebra over a field* F*. Let* H *be an ideal of* L *having finite codimension. Then* H *is finitely generated as an ideal.*

Proof — Let

$$M = \{a_1, \ldots, a_n\}$$

be a finite subset generated L, and let B be a subspace of L such that $L = B \oplus H$. Let $\mathrm{codim}_F(H) = d$. Then $\dim_F(B) = d$. Choose in B some basis $\{b_1, \ldots, b_d\}$. Denote by pr_B (respectively pr_H) the canonical projection of L on B (respectively H). Let E be the ideal, generated by the elements

$$\{pr_H(a_j), pr_H([a_j, b_m]), pr_H([b_m, a_j]) | 1 \leqslant j \leqslant n, 1 \leqslant m \leqslant d\}.$$

By such choice H includes E, and E is a finitely generated as an ideal of L. If x is an arbitrary element of $E + B$, then $x = u + b$ where $u \in E$ and $b \in B$. Furthermore

$$b = \alpha_1 b_1 + \ldots + \alpha_d b_d$$

for suitable elements $\alpha_1, \ldots, \alpha_d \in F$. We have

$$[b, a_j] = [\alpha_1 b_1 + \ldots + \alpha_d b_d, a_j] = \alpha_1 [b_1, a_j] + \ldots + \alpha_d [b_d, a_j] =$$

$$\alpha_1(pr_H([b_1, a_j]) + pr_B([b_1, a_j]) + \ldots + \alpha_d(pr_H([b_d, a_j]) + pr_B([b_d, a_j]) =$$

$$\alpha_1 pr_H([b_1, a_j]) + \ldots + \alpha_d pr_H([b_d, a_j]) + \alpha_1 pr_B([b_1, a_j]) + \ldots + \alpha_d pr_B([b_d, a_j]);$$

$$[a_j, b] = [a_j, \alpha_1 b_1 + \ldots + \alpha_d b_d] = \alpha_1 [a_j, b_1] + \ldots + \alpha_d [a_j, b_d] =$$

$$\alpha_1 (pr_H([a_j, b_1]) + pr_B([a_j, b_1]) + \ldots + \alpha_d (pr_H([a_j, b_d]) + pr_B([a_j, b_d]) =$$

$$\alpha_1 pr_H([a_j, b_1]) + \ldots + \alpha_d pr_H([a_j, b_d]) + \alpha_1 pr_B([a_j, b_1]) + \ldots + \alpha_d pr_B([a_j, b_d]).$$

The elements

$$\Sigma 1 \leqslant m \leqslant d(\alpha_m pr_H([b_m, a_j]) + \alpha_m pr_B([b_m, a_j])),$$

and
$$\Sigma 1 \leqslant m \leqslant d(\alpha_m pr_H([a_j, b_m]) + \alpha_m pr_B([a_j, b_m]))$$
clearly belong to $E + B$. It follows that $E + B$ is an ideal of A. Since
$$a_j = pr_H(a_j) + pr_B(a_j) \in E + B, 1 \leqslant j \leqslant n,$$
then
$$E + B = A = H + B.$$
The inclusion $E \leqslant H$ and the equation $H \cap B = \langle 0 \rangle$ imply that $H = E$. In particular, H is a finitely generated as an ideal. □

Corollary 2.2 *Let* L *be a finitely generated Leibniz algebra over a field* F*. If* L *is nilpotent, then* L *has finite dimension.*

Proof — Let
$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_n = L$$
be the upper central series of L. Proposition 2.1 shows that Z_{n-1} is finitely generated as an ideal, since L/Z_{n-1} is abelian and the dimension $\dim_F(L/\zeta_{n-1}(L))$ is finite. The inclusion
$$Z_{n-1}/Z_{n-2} \leqslant \zeta(L/Z_{n-2})$$
implies that Z_{n-1}/Z_{n-2} is finitely generated as a subalgebra. In turn out, it follows that $\dim_F(Z_{n-1}/Z_{n-2})$ is finite. Then $\dim_F(L/Z_{n-2})$ is finite. Using the similar arguments and ordinary induction we prove that $\dim_F(L)$ is finite. □

Proof of Theorem A — Let
$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_\alpha \leqslant Z_{\alpha+1} \leqslant \ldots Z_\gamma = \zeta_\infty(L) = L$$
be the upper central series of L. Since L is finitely generated, γ is not a limit ordinal. Suppose that γ is infinite, then $\gamma = \kappa + n$ for some limit ordinal $\kappa \geqslant \omega$. Then L/Z_κ is a nilpotent finitely generated Leibniz algebra, and Corollary 2.2 shows that L/Z has finite dimension. Then Proposition 2.1 implies that Z_κ is finitely generated as an ideal. Let
$$W = \{w_1, \ldots, w_m\}$$
be a finite subset such that Z_κ is generated by W as ideal. From the