Chapter 7

Stability and control, introduction to helicopter flight dynamics

7.1. Introduction

The properties analyzed in this chapter are concerned with the response of the helicopter after the perturbation of a steady trimmed flight condition, produced by the action of a gust or the action of the pilot through flight controls.

In particular, helicopter behavior is expressed in terms of stability and control characteristics, which configure the *flight qualities*; these topics constitute a significant part of the flight dynamics.

This chapter introduces some fundamental problems of helicopter stability and control by means of theories using typical assumptions to simplify the approach.

Therefore, as in the basic analysis of fixed-wing aircraft, we assume the following for the disturbed motion of the helicopter: small disturbances and the separation of longitudinal and lateral motions. For the latter case, we saw that its consequences represent major critical issues for the analysis applied to the helicopter (remember the natural mating between the two types of motion due to the modalities of main rotor flapping). For a conventional helicopter configuration with a single main rotor, which we will analyze in this chapter, the tail rotor confers asymmetry to the whole rotorcraft, which requires solving all the equations of motion simultaneously, for a rigorous approach.

However, it is general practice to set up basic analysis on the separation of the two types of motions, for the following reasons: considerable problem simplification and interesting obtained results. Therefore, the treatment that follows adopts the assumptions above.

Finally, the arguments incorporate methodologies and procedures ready to be implemented on the computer.

7.2. The single-degree of freedom dynamic system

Before introducing the helicopter stability, it is very useful to review the properties of the system composed of a mass, a spring and a damper, that can be modelled by a second-order differential equation. This system can be used to understand and to represent many dynamic systems, and it provides results which are needed for the presentation of the arguments that follow.
Thus, in the general model (shown in Figure 7.1) a force $F(t)$, that is the forcing function or the applied force, acts on the mass m; in x-direction, there are also a linear force provided by the spring and a damping force, proportional to the mass velocity, provided by the damper.

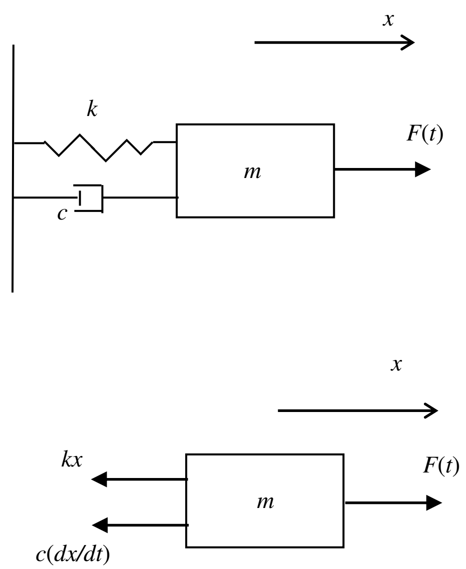

Figure 7.1 Mass/spring/damper dynamic system, single-degree of freedom

Homogeneous solution or free response

Considering that $m(dx^2/dt)$ is the inertia force, the following second-order differential equation describes the dynamic system shown in Figure 7.1:

$$m\frac{dx^2}{dt}+c\frac{dx}{dt}+kx=F(t)$$

It is an ordinary differential equation with constant coefficients.
The solution of the homogeneous equation

$$m\frac{dx^2}{dt}+c\frac{dx}{dt}+kx=0$$

provides the transient or free response of the system. The solution is found by substituting $x = A\mathrm{e}^{\lambda t}$ into the equation; therefore, we obtain:

$$\lambda^2+\left(\frac{c}{m}\right)\lambda+\left(\frac{k}{m}\right)=0$$

which has the following roots:

$$\lambda_{1,2}=-\left(\frac{c}{2m}\right)\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}$$

Therefore, the solution of the homogeneous differential equation is:

$$x(t)=a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}$$

$$x(t)=a_1e^{\left[-\left(\frac{c}{2m}\right)+\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}\right]t}+a_2e^{\left[-\left(\frac{c}{2m}\right)-\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}\right]t}$$

and it represents the free response of the damped system, where a_1 and a_2 are constants and are determined from the initial conditions. This solution depends on the values of m, c and k. In particular, consider that if we have

$$\left(\frac{c}{2m}\right)<\sqrt{\left(\frac{k}{m}\right)}$$

the solution is

$$x(t)=a_1 e^{\left[-\left(\frac{c}{2m}\right)+i\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c}{2m}\right)^2}\right]t}+a_2 e^{\left[-\left(\frac{c}{2m}\right)-i\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c}{2m}\right)^2}-\right]t}$$

$$x(t)=e^{\left(-\frac{c}{2m}\right)t}\left[A_1\cos\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c}{2m}\right)^2}\cdot t+A_2\sin\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c}{2m}\right)^2}\cdot t\right]$$

This solution describes a damped sinusoidal motion, characterized by the following *damped natural frequency* ω:

$$\omega=\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c}{2m}\right)^2}$$

Consider that if we have

$$\left(\frac{c}{2m}\right)=\sqrt{\left(\frac{k}{m}\right)}$$

the solution $x(t)$ describes a critical damped motion; in this case, we have

$$c_{cr}=2\sqrt{\left(\frac{m^2k}{m}\right)}=2\sqrt{km}$$

where c_{cr} is defined as the *critical damping constant*, and the ratio ζ

$$\xi=\frac{c}{c_{cr}}$$

is defined as the *damping ratio*.
Now, let us write the homogeneous equation for the undamped system (c=0):

$$m\frac{dx^2}{dt}+kx=0$$

By using the previous procedure, we obtain the following solution:

$$x(t)=\left[C\cos\sqrt{\left(\frac{k}{m}\right)}\cdot t+D\sin\sqrt{\left(\frac{k}{m}\right)}\cdot t\right]=E\left[\cos\sqrt{\left(\frac{k}{m}\right)}\cdot t+\varphi\right]$$

which describes a steady sinusoidal motion, characterized by the following *undamped natural frequency* ω_n:

$$\omega_n=\sqrt{\left(\frac{k}{m}\right)}$$

Finally, Figure 7.2 shows all the solutions as functions of *m*, *c* and *k*.

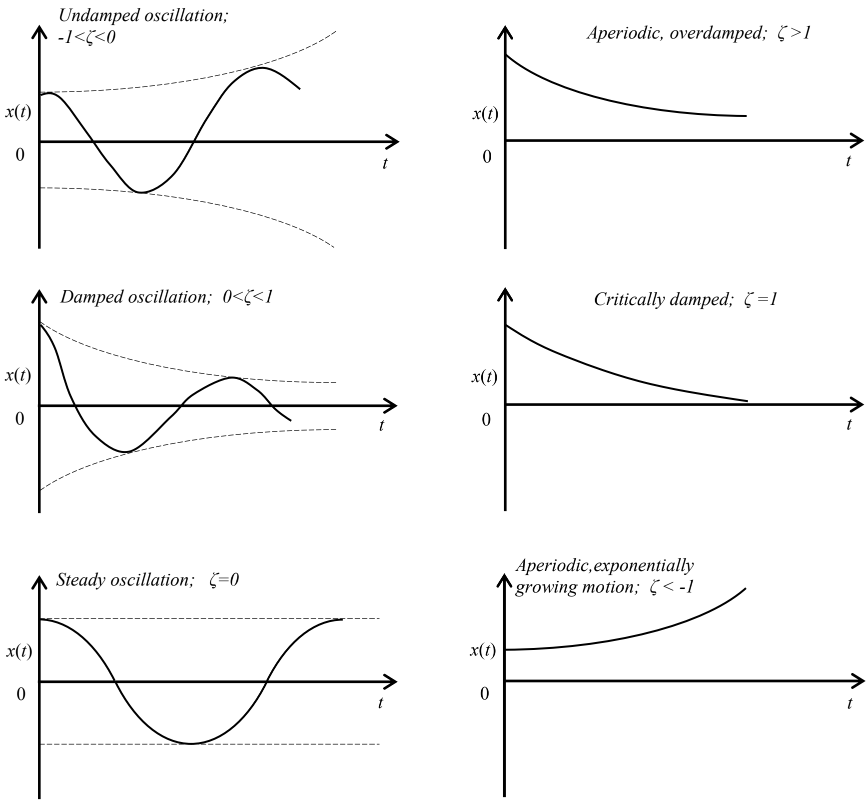

Figure 7.2 Types of free response of the dynamic system with a single-degree of freedom

Finally, using the parameters defined above, the second-order differential equation with constant coefficients that describes the mass/spring/damper dynamic system shown in Figure 7.1 can be written as:

$$\frac{dx^2}{dt} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = \frac{1}{m} F(t)$$

Therefore, the damped natural frequency ω, the damping ratio ζ and un-damped natural frequency ω_n are determined from the analysis of the free response of the system. In fact, note that the solution of the following characteristic equation

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

can be written, in a general form, as

$$\lambda_{1,2} = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2}$$

Particular solution corresponding to a sinusoidal applied force

Now, let us consider the case where the forcing function $F(t) \neq 0$

$$\frac{dx^2}{dt} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = \frac{1}{m} F(t)$$

and is equal to $F(t)/\mathrm{m} = F_0 \cos\omega t$. Therefore, the equation of the dynamic system (with a single-degree of freedom) becomes:

$$\frac{dx^2}{dt} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = F_0 \cos\omega t$$

Before continuing, let us remember that the solution of the second-order differential equation is the sum of the solution of the homogeneous equation, that represents the transient motion, with $F(t)=0$, and of a particular solution of the complete equation, the steady motion, with $F(t) \neq 0$.
Hence:

$$x(t) = [x(t)]_{homogeneous\ eq} + [x(t)]_{particular\ solution}$$

We have that: $\{[x(t)]_{particular\ solution} = X_f \cos(\omega_f t + \phi)\}$, where the *response amplitude* X_f and the *phase angle* ϕ are given by the following expressions:

$$X_f = \frac{F_0}{m}\frac{1}{\sqrt{\left(\omega_n^2 - \omega_f^2\right)^2 + 4\zeta^2\omega_n^2\omega_f^2}}, \qquad \phi = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega_f}{\omega_n^2 - \omega_f^2}\right)$$

These relations define the frequency response of the system.

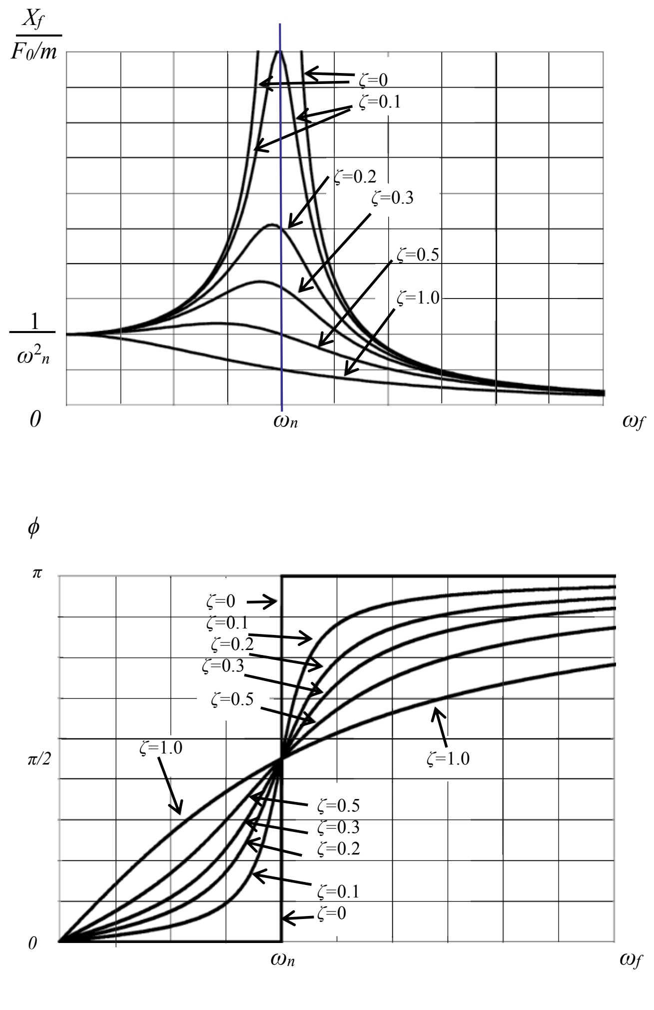

Figure 7.3 Amplitude and phase, frequency response

Transfer function of the mass/spring/damper system
Considering that the equation of the system is

$$\frac{dx^2}{dt} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = \frac{1}{m} F(t) = f(t)$$

then, let us write

$$x(t) = x_1(t)$$
$$\frac{dx}{dt} = \dot{x} = x_2(t)$$

Therefore, we obtain

$$\frac{dx_1}{dt} = \dot{x}_1 = x_2(t)$$
$$\frac{dx_2}{dt} + 2\zeta\omega_n x_2 + \omega_n^2 x_1 = f(t)$$

Choose the initial conditions as

$$x(0) = 0$$
$$\frac{dx(0)}{dt} = 0$$

write the Laplace transform of $x(t)$ and of $f(t)$:

$$\mathfrak{L}[x_1(t)] = Y(s)$$

and

$$\mathfrak{L}\left[\frac{dx_2}{dt} + 2\zeta\omega_n x_2 + \omega_n^2 x_1\right] = \mathfrak{L}\left[\frac{dx_2}{dt}\right] + 2\zeta\omega_n\, \mathfrak{L}[x_2(t)] + \omega^2{}_n\, \mathfrak{L}[x_1(t)] = U(s)$$

From the relations above we have:

$$s^2 Y(s) + 2\zeta\omega_n\, sY(s) + \omega^2{}_n\, Y(s) = U(s)$$

Finally, the transfer function $G(s)$ of the system, that is the ratio of the output and the input, is equal to:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+2\zeta\omega_n+\omega_n^2}$$

State-space modeling
The following relations

$$\dot{x}_1=x_2(t)$$
$$\dot{x}_2+2\zeta\omega_n x_2+\omega_n^2 x_1=f(t)$$

can be written in matrix form as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

where $u(t)$=$f(t)$. We have:

$$\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Bu}$$

with

$$\dot{\mathbf{x}}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix},\qquad \mathbf{A}=\begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

and $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is the state vector.
The system is fully described by the state-space matrices **A** and **B**.
Now, we know that the free response of the system, where $f(t)$=0, may be studied by the equations:

$$\dot{\mathbf{x}}=\mathbf{Ax}$$

The substitution of $\mathbf{x}(t)=\mathbf{x}_j e^{\lambda_j t}$ into equations above gives

$$(\mathbf{A}-\lambda_j\mathbf{I})\mathbf{x}_j=0 \qquad \text{where } \mathbf{I} \text{ is the identity matrix.}$$

Now, the vector $\mathbf{x}_j$ is the eigenvector associated with the eigenvalue λ_j of the matrix **A**. The solution is the following linear combination:

$$\mathbf{x}(t) = \sum_{j=1}^{2} c_j \mathbf{x}_j e^{\lambda_j t}$$

(c_j is a constant that is fixed by the initial conditions)

Control form of a second-order differential equation
If the system has mass m=1, then it can be visualized by the diagram in Figure 7.4:

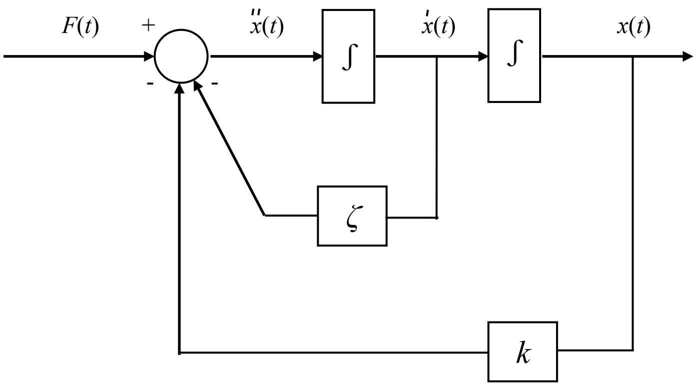

Figure 7.4 Control form of the second-order differential equation

7.3. Helicopter static stability and dynamic stability

The stability, in general terms, is defined as the capability to restore an initial trim condition that has been perturbed by a particular cause.

Static stability is defined as the initial tendency of the system to return to the trim condition. Then, *dynamic stability* is defined as the tendency of the system to restore the trim condition as the time goes on. In other words, the static stability studies the initial motion (initial response) of the aircraft after the perturbation. Instead, the dynamic stability is concerned with the evolution of the aircraft motion versus time, in relation with the tendency to return to or to leave the trim condition that has been perturbed.

It should be noted that an aircraft can be statically stable but dynamically unstable. However, the static stability is a necessary condition but is not a sufficient condition for the dynamic stability.

7.4. Helicopter static stability

In the pages that follow we will discuss some fundamental cases related especially to the main rotor properties, because it supplies a relevant contribution to the stability characteristics of the helicopter as a whole.

7.4.1. *Stability following forward speed perturbation*

In the context of the aircraft response immediately following a disturbance, as first case, we treat the response to speed perturbation in the direction of the motion. Supposing to analyze a forward flight condition, for the reason we saw in the previous chapters, an increase in forward speed will involve an increase of the rotor flapping with backward inclination of rotor disc. Therefore, the rotor thrust is characterized by a component in the tail direction that opposes the disturbance: the rotor supplies a contribution to the static stability. The fuselage, instead, can provide a contribution to stability or a contribution to instability, depending on the direction of the generated aerodynamic forces (lift and drag). It is also clear that an additional contribution to stability can be provided by the horizontal stabilizer, depending on its dimensions and position on the entire helicopter. These considerations are valid for both forward and hovering flight, taking into account the fact that as the speed decreases the contribution from the fuselage and from the horizontal tail tends to decrease (until being negligible in estimation at very low flight speed).

7.4.2. *Stability following vertical speed or incidence perturbation*

Assume a steady level flight condition; as a consequence of a vertical gust, the main rotor blades have an increase in incidence and the rotor thrust also increases. The total effect on the advancing and retreating blades (considering also the difference in relative speed) produces backward flapping of the rotor, with the generation of a nose up pitching moment. Indeed, after the inclination of the rotor disc, this moment is due to the thrust: the rotor is statically unstable. It is clear that the rotor instability grows as the forward speed increases. The considerations discussed above for the fuselage are still valid, but generally its contribution is in terms of instability. The only one contribution to stability is provided by the horizontal stabilizer: this contribution grows as the forward speed increases. Finally, note that the availability of accurate methods for analysis of aeroelastic phenomena of rotor blades and the use of advanced composite materials can allow the designer to obtain appropriate load distributions to contain the unstable effect of the rotor on the response to the incidence perturbation.

7.4.3. *Stability following yawing perturbation*

Assuming an attitude with a yawing angle different from zero, a change in the incidence of the tail rotor is obtained; this variation provides a damping effect, similar to that of a vertical fin (known as '*fin effect*'). Therefore, the vertical fin provides an important contribution to the stability, because it responds generating a lateral force that produces a consequent yawing moment. This moment confers stability.

However, different from fixed-wing aircraft, it shall be noted that evaluation regarding the fin (and the tail rotor) of the helicopter shall take into account the effects due to the main rotor wake on the empennage. Increasing the forward speed, we shall consider also the contribution from the fuselage (generally neglected at low flight speed): its action can be of stable or unstable type, depending on the disturbance, the position of the centre of gravity and the airframe geometry.

7.5. Helicopter dynamic stability

Procedures and methods for the helicopter stability analysis (from the determination of equations of motion to the application to the flight test issues) have been developed, since early studies, by extending the methodologies applied to the fixed-wing aircraft. Therefore, also the topic that follows is characterized by an initial formal setting clearly common for rotary-wing and fixed-wing aircraft, and by a subsequent stage, where we find the typical problems to be solved for the helicopter.

Again, we recall Equations (5.5b) determined in Chapter 5, with the rigid body assumption. Now, for an accurate assessment, each rotor blade has its own degree of freedom, which provides a contribution to the perturbed motion. Consequently, for the helicopter with a single main rotor, over the six equations of motion as for a fixed-wing aircraft (three for the translation and three for the rotation about the reference axes) we shall add, in a basic approach, as minimum other three equations: one for rotor longitudinal flapping, one for lateral flapping, and one for conic attitude of the rotating blades. Remember that due to rapid response of the blades versus the whole airframe, *the rotor can be assumed as a compact generator of forces and moments,* neglecting the motion of the single blade. Then, this is a quasi-static condition of motion, by which, now, the equations are only six (because there are six active degrees of freedom). Obviously, this condition cannot be maintained in those cases where the designer needs to study aeroelastic phenomena or resonance. However, those cases are beyond the purposes of the present book. From the previous notes and assumptions, we recall the set of Equations (5.5b), now with the new number (7.1):

$$\begin{cases} \frac{W_G}{g}(\dot{V}_x + \omega_y V_z - \omega_z V_y) = F_{xa} + W_x \\ \frac{W_G}{g}(\dot{V}_y - \omega_x V_z + \omega_z V_x) = F_{ya} + W_y \\ \frac{W_G}{g}(\dot{V}_z + \omega_x V_y - \omega_y V_x) = F_{za} + W_z \\ \dot{\omega}_x I_x - \omega_y \omega_z (I_y - I_z) - (\dot{\omega}_z + \omega_x \omega_y) I_{xz} = M_{xa} \\ \dot{\omega}_y I_y - \omega_x \omega_z (I_z - I_x) - (\omega_z^2 - \omega_x^2) I_{xz} = M_{ya} \\ \dot{\omega}_z I_z - \omega_x \omega_y (I_x - I_y) - (\dot{\omega}_x - \omega_y \omega_z) I_{xz} = M_{za} \end{cases} \tag{7.1}$$

with the auxiliary Equations (5.5b):

$$\dot{\Phi} = \omega_x + \omega_y \sin\Phi tg\Theta + \omega_z \cos\Phi tg\Theta$$
$$\dot{\Theta} = \omega_y \cos\Phi - \omega_z \sin\Phi$$
$$\dot{\Psi} = \omega_y \sin\Phi \sec\Theta + \omega_z \cos\Phi \sec\Theta$$

Equations (7.1), representing the balance of forces and moments acting on the helicopter, constitute the basic model to study the aircraft motion. We remember that the equations have been written with respect to the body axes, with the following assumptions: rigid body, constant mass, existence of the aircraft plane of symmetry.

From a general point of view, each Equation (7.1) is related to a degree of freedom, having therefore three degrees of freedom for translation and three degrees of freedom for rotation. Going in depth into dynamic stability analysis, we will discuss the fixed control cases. The classical approach (that is followed in the present book) considers that the action by a gust or the action to perform a manoeuvre generate an unsteady flight condition, analyzed superimposing a *'disturbance'* to the initial steady condition of motion. From a mathematical point of view, this approach requires to consider the terms in the set (7.1) equal to the sum of the value in the initial steady trim condition and the value of perturbation. Therefore, we can write:

$$
\begin{aligned}
&V_x = V_{x0} + u \qquad && V_y = V_{y0} + v \qquad && V_z = V_{z0} + w \\
&\omega_x = p_0 + p && \omega_y = q_0 + q && \omega_z = r_0 + r \\
&F_{xa} = F_{x0} + X && F_{ya} = F_{y0} + Y && F_{za} = F_{z0} + Z \\
&M_{xa} = M_{x0} + L && M_{ya} = M_{y0} + M && M_{za} = M_{z0} + N \\
&\Theta = \Theta_0 + \Theta_d && \Phi = \Phi_0 + \Phi_d && \Psi = \Psi_0 + \Psi_d
\end{aligned}
\tag{7.2}
$$

where the terms with subscript '0' represent the initial condition of motion and the second term (we take, for example, u) defines the disturbance.

By using expressions (7.2), the first equation of the set (7.1) can be written in the following form:

$$
\frac{W_G}{g}[(\dot{V}_{xo} + \dot{u}) + (q_0 + q)(V_{z0} + w) - (r_0 + r)(V_{y0} + v)] = (F_{x0} + X) + \\
- W_G \sin(\Theta_0 + \Theta_d)
$$

and,

$$
\frac{W}{g}[(\dot{V}_{xo} + \dot{u}) + (q_0 V_{z0} + q_0 w + q V_{z0} + q w) - (r_0 V_{y0} + r V_{y0} + r_0 v + r v)] = \\
= (F_{x0} + X) - W \sin(\Theta_0 + \Theta_d) \tag{7.3}
$$

Now, consider that in the trim condition, before introducing the disturbance, we can write the following expression:

$$
\frac{W_G}{g}(\dot{V}_{x0} + q_0 V_{z0} - r_0 V_{y0}) = F_{x0} - W_G \sin\Theta_0
$$

Therefore, the Equation (7.3) can be written as:

$$\frac{W}{g}(\dot{u}+qV_{z0}+q_0w+qw-rV_{y0}-r_0v-rv)=X-W[\sin(\Theta_0+\Theta_d)-\sin\Theta_0]$$

Operating in a similar manner on the other five equations and using the expressions (7.2), finally, we obtain the following set of equations:

$$\begin{cases}\frac{W_G}{g}(\dot{u}+qw-rv+V_{z0}q+q_0w-V_{y0}r-vr_0)=X-W_G[\sin(\Theta_0+\Theta_d)-\sin\Theta_0] \\ \frac{W_G}{g}(\dot{v}+ur-wp+V_{x0}r+r_0u-V_{z0}p-p_0w)=Y+W_G[\cos(\Theta_0+\Theta_d)\sin(\Phi_0+\Phi_d)+ \\ \qquad -\cos\Theta_0\sin\Phi_0] \\ \frac{W_G}{g}(\dot{w}+vp-uq+V_{y0}p+vp_0-V_{x0}q-q_0u)=Z+W_G[\cos(\Theta_0+\Theta_d)\cos(\Phi_0+\Phi_d)+ \\ \qquad -\cos\Theta_0\cos\Phi_0] \\ \dot{p}I_x-(q_0r+r_0q+qr)(I_y-I_z)-(\dot{r}+p_0q+q_0p+pq)I_{xz}=L \\ \dot{q}I_y-(p_0r+r_0p+pr)(I_z-I_x)-(2r_0r-2p_0p-p^2+r^2)I_{xz}=M \\ \dot{r}I_z-(p_0q+q_0p+pq)(I_x-I_y)-(\dot{p}-q_0r-r_0q-qr)I_{xz}=N \end{cases} \tag{7.4}$$

The system (7.4) is the set of equations for the perturbed motion in the general form.

7.5.1. *Small disturbance theory*

The perturbed dynamics is based on the resolution of the set of Equations (7.4), once the initial condition of motion has been fixed and the forces and moments acting on the aircraft have been defined for each scalar equation. In order to approach this typology of problem, methods and assumptions shall be obviously defined in accordance with the task to be accomplished. Considering the notes discussed in the previous pages, *the small disturbance assumption* can be valid and applicable to many problems and constitutes the initial approach for many dynamic analyses due to the simplification of the mathematical models and to the interesting results which can be obtained.

From a mathematical standpoint, this assumption allows the disturbance quantities and their derivatives to be considered small, and, consequently, we can neglect their products and squares into Equations (7.4). Moreover, angles are considered so small that the cosine can be considered equal to 1, and the sine and the tangent equal to the value of the angle expressed in radians. Therefore, let us use the following expressions:

$$\sin(\Theta_0 + \Theta_d) \approx \sin\Theta_0 + \Theta_d \cos\Theta_0$$
$$\cos(\Theta_0 + \Theta_d) \approx \cos\Theta_0 - \Theta_d \sin\Theta_0$$

Now, Equations (7.4) become:

$$\begin{cases} \frac{W_G}{g}(\dot{u} + V_{z0}q + q_0 w - V_{y0}r - r_0 v) = X - W_G[\Theta_d \cos\Theta_0] \\ \frac{W_G}{g}(\dot{v} + V_{x0}r + r_0 u - V_{z0}p - p_0 w) = Y + W_G[\Phi_d \cos\Phi_0 \cos\Theta_0 - \Theta_d \sin\Theta_0 \sin\Phi_0] \\ \frac{W_G}{g}(\dot{w} + V_{y0}p + p_0 v - V_{x0}q - q_0 u) = Z - W_G(\Phi_d \cos\Theta_0 \sin\Phi_0 + \Theta_d \sin\Theta_0 \cos\Phi_0) \\ \dot{p}I_x - (q_0 r + r_0 q)(I_y - I_z) - (\dot{r} + p_0 q + q_0 p)I_{xz} = L \\ \dot{q}I_y - (p_0 r + r_0 p)(I_z - I_x) - (2r_0 r - 2p_0 p)I_{xz} = M \\ \dot{r}I_z - (p_0 q + q_0 p)(I_x - I_y) - (\dot{p} - q_0 r - r_0 q)I_{xz} = N \end{cases} \tag{7.5}$$

Assuming that the initial trim condition is, for formal convenience, a steady level flight condition with constant speed $\mathbf{V}(V_{x0}, 0, V_{z0})$, we may write

$$V_{y0} = p_0 = q_0 = r_0 = 0, \qquad \text{and also } \Psi_0 = \Phi_0 = 0$$

and the set of Equations (7.5) becomes:

$$\begin{cases} \frac{W_G}{g}(\dot{u} + V_{z0}q) = X - W_G[\Theta_d \cos\Theta_0] \\ \frac{W_G}{g}(\dot{v} + V_{x0}r - V_{z0}p) = Y + W_G[\Phi_d \cos\Theta_0] \\ \frac{W_G}{g}(\dot{w} - V_{x0}q) = Z - W_G(\Theta_d \sin\Theta_0) \\ \dot{p}I_x - \dot{r}I_{xz} = L \\ \dot{q}I_y = M \\ \dot{r}I_z - \dot{p}I_{xz} = N \end{cases} \tag{7.6}$$

Instead, if the initial trim condition is a *hovering flight* condition, we may write

$$V_{x0} = V_{y0} = V_{zo} = p_0 = q_0 = r_0 = 0$$

and

$$\Psi_0 = \Theta_0 = \Phi_0 = 0$$

finally, obtaining:

$$\begin{cases} \dfrac{W_G}{g}(\dot{u}) = X - W_G[\Theta_d] \\ \dfrac{W_G}{g}(\dot{v}) = Y + W_G[\Phi_d] \\ \dfrac{W_G}{g}(\dot{w}) = Z \\ \dot{p}I_x - \dot{r}I_{xz} = L \\ \dot{q}I_y = M \\ \dot{r}I_z - \dot{p}I_{xz} = N \end{cases} \tag{7.7}$$

7.5.2. *Stability derivatives*

The parameters representing the perturbation from trim values are written using a Taylor series with the first terms only (linear terms, small disturbance assumption). Then, the expression for the force increment *X* is:

$$X = \frac{\partial X}{\partial u}u + \frac{\partial X}{\partial v}v + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial p}p + \frac{\partial X}{\partial q}q + \frac{\partial X}{\partial r}r + \frac{\partial X}{\partial \theta_{RP}}\theta_{MR} + \frac{\partial X}{\partial A_1}A_1 + \\ + \frac{\partial X}{\partial B_1}B_1 + \frac{\partial X}{\partial \theta_{tr}}\theta_{tr}$$

In the expression above we find also the terms θ_{MR}, B_1, A_1, θ_{tr} (that are, in this chapter, variations from trim values) associated respectively with the following control inputs: collective pitch, longitudinal cyclic pitch and lateral cyclic pitch of the main rotor, and collective pitch of the tail rotor.
In order to simplify the notation, we adopt the following compact form:

$$X = X_u u + X_v v + X_w w + X_p p + X_q q + X_r r + X_{\theta_{MR}}\theta_{MR} + X_{A_1}A_1 + X_{B_1}B_1 + \\ + X_{\theta_{tr}}\theta_{tr}$$

where the generic derivative has been written as $\partial X/\partial a = X_a$.

Using the same procedure for each term representing an increment, finally we obtain the set of expressions:

$$F_{xa} = F_{x0} + X_u u + X_v v + X_w w + X_p p + X_q q + X_r r + X_{\theta_{MR}} \theta_{MR} + X_{A_1} A_1 + X_{B_1} B_1 + X_{\theta_{tr}} \theta_{tr}$$

$$F_{ya} = F_{y0} + Y_u u + Y_v v + Y_w w + Y_p p + Y_q q + Y_r r + Y_{\theta_{MR}} \theta_{MR} + Y_{A_1} A_1 + Y_{B_1} B_1 + Y_{\theta_{tr}} \theta_{tr}$$

$$F_{za} = F_{z0} + Z_u u + Z_v v + Z_w w + Z_p p + Z_q q + Z_r r + Z_{\theta_{MR}} \theta_{MR} + Z_{A_1} A_1 + Z_{B_1} B_1 + Z_{\theta_{tr}} \theta_{tr} + Z_{\dot{w}} \dot{w}$$

$$M_{xa} = M_{x0} + L_u u + L_v v + L_w w + L_p p + L_q q + L_r r + L_{\theta_{MR}} \theta_{MR} + L_{A_1} A_1 + L_{B_1} B_1 + L_{\theta_{tr}} \theta_{tr}$$

$$M_{ya} = M_{y0} + M_u u + M_v v + M_w w + M_p p + M_q q + M_r r + M_{\theta_{MR}} \theta_{MR} + M_{A_1} A_1 + M_{B_1} B_1 + M_{\theta_{tr}} \theta_{tr} + M_{\dot{w}} \dot{w}$$

$$M_{za} = M_{z0} + N_u u + N_v v + N_w w + N_p p + N_q q + N_r r + N_{\theta_{MR}} \theta_{MR} + N_{A_1} A_1 + N_{B_1} B_1 + N_{\theta_{tr}} \theta_{tr}$$

As in fixed-wing aircraft analysis, the derivatives in u, v, w, p, q, r contained in the expressions above are called *stability derivatives*, and the derivatives in θ_{MR}, B_1, A_1, θ_{tr} are called *control derivatives*.

The derivatives are expressed in a so called normalized form when those related to the forces are divided by the mass M_{heli} of the helicopter, and those related to the moments are divided by an appropriate moment of inertia.

Note that the third and the fifth expressions contain also the derivatives $\partial Z/\partial \dot{w}$ and $\partial M/\partial \dot{w}$, related to change of force along the Z-axis and to change of moment about the Y-axis respectively, due to the acceleration $\dot{w}$ (as in fixed-wing aircraft analysis); in the expressions above, they are the contributions due to an acceleration, remembering that we are considering the disturbances calculated as functions of speed. In particular, their contributions take into account the downwash effect of the main rotor on the horizontal stabilizer. Therefore, $\partial Z/\partial \dot{w}$ and $\partial M/\partial \dot{w}$ are kept when a sophisticated investigation is required to perform the analysis; generally, they are neglected in order to simplify the treatment.

The stability derivatives are evaluated in the steady trim conditions; moreover, the derivatives are constant. Generally, in hovering flight each derivative is obtained by determining the contributions due to the main rotor and to the tail rotor. In forward flight, the designer shall consider also the contributions due to the fuselage, to the horizontal stabilizer and to the vertical fin. The derivatives may be determined by using various procedures. The analytical or classical methodology (appropriate for an initial analysis) requires to write the equations of forces and moments for the rotors, for the fuselage and empennage (that we wrote in detail in Chapter 5, Section 5.6 and 5.8) and then to apply the derivative operation. Moreover, from the relations obtained in Chapter 5, we know that the rotor force and moment derivatives are directly related to the thrust and flapping derivatives.

7.5.2.1. *Force perturbation expressions and stability derivatives*

Thus, considering the parameters in Figures 5.2b-5.5 and the relations that we wrote in detail in Chapter 5, Section 5.8, the force increments X, Y, Z along the body axes, that we find into relations (7.2), may be expressed as:

$$X = -T_{TPP}\Delta a_{1s} - a_{1s}\Delta T_{TPP} - H_{TPP} - \Delta X_{Fuselage+Tail\ empennage}$$

$$Y = T_{TPP}\Delta b_{1s} + b_{1s}\Delta T_{TPP} + \Delta Y_{Fuselage+Tail\ empennage} + \Delta T_{tr}$$

$$Z = -\Delta T_{TPP}$$

Therefore, for X_u, X_w, X_q we have:

$$\frac{\partial X}{\partial u} = X_u = -T_{TPP}\frac{\partial a_{1S}}{\partial u} - a_{1S}\frac{\partial T_{TPP}}{\partial u} - \frac{\partial H_{TPP}}{\partial u} - \frac{\partial X_{Fus+tail\_emp}}{\partial u}$$

$$\frac{\partial X}{\partial w} = X_w = -T_{TPP}\frac{\partial a_{1S}}{\partial w} - a_{1S}\frac{\partial T_{TPP}}{\partial w} - \frac{\partial H_{TPP}}{\partial w} - \frac{\partial X_{Fus+tail\_emp}}{\partial w}$$

$$\frac{\partial X}{\partial q} = X_q = -T_{TPP}\frac{\partial a_{1S}}{\partial q} - a_{1S}\frac{\partial T_{TPP}}{\partial q} - \frac{\partial H_{TPP}}{\partial q} - \frac{\partial X_{Fus+tail\_emp}}{\partial q}$$

and for Z_u, Z_w and Z_q we have:

$$\frac{\partial Z}{\partial u} = Z_u = -\frac{\partial T_{TPP}}{\partial u}$$

$$\frac{\partial Z}{\partial w} = Z_w = -\frac{\partial T_{TPP}}{\partial w}$$

$$\frac{\partial Z}{\partial q} = Z_q = -\frac{\partial T_{TPP}}{\partial q}$$

7.5.2.2. *Moment perturbation expressions and stability derivatives*

Considering the parameters in Figures 5.2b-5.5 and the relationships that we wrote for the moments about the body axes in Chapter 5, Section 5.8, we obtain the expressions for the moment increments (or change in moments) L, M, N:

$$L = (L_{MR}\,\Delta b_{1s}) + (T_{TPP}\,h_z\,\Delta b_{1s}) + (h_z\,b_{1s}\,\Delta T_{TPP}) + (\Delta L_{Fuselage+Tail\ empennage}) + (z_{tr}\,\Delta T_{tr})$$

$$M = (M_{MR}\,\Delta a_{1s}) + (T_{TPP}\,h_z\,\Delta a_{1s}) + (h_z\,a_{1s}\,\Delta T_{TPP}) - (h_x\,\Delta T_{TPP}) + (h_z\,\Delta H_{TPP}) + (\Delta M_{Fuselage+Tail\ empennage})$$

$$N = -(T_{TPP}\,h_x\,\Delta b_{1s}) - (h_x\,b_{1s}\,\Delta T_{TPP}) + (\Delta N_{Fuselage+Tail\ empennage}) + (l_{tr}\Delta T_{tr}) + (\Delta Q)$$

where $L_{MR} = \frac{dL_{MR}}{db_{1S}} b_{1S}$, and $M_{MR} = \frac{dM_{MR}}{da_{1S}} a_{1S}$ (see Chapter 3, Sect.3.4).

Therefore, for L_v, L_r we have:

$$\frac{\partial L}{\partial v} = L_v = L_{MR}\frac{\partial b_{1S}}{\partial v} + h_z T_{TPP}\frac{\partial b_{1s}}{\partial v} + h_z b_{1S}\frac{\partial T_{TPP}}{\partial v} + \frac{\partial L_{Fus+tail\_emp}}{\partial v} + z_{tr}\frac{\partial T_{tr}}{\partial v}$$

$$\frac{\partial L}{\partial r} = L_r = L_{MR}\frac{\partial b_{1S}}{\partial r} + h_z T_{TPP}\frac{\partial b_{1s}}{\partial r} + h_z b_{1S}\frac{\partial T_{TPP}}{\partial r} + \frac{\partial L_{Fus+tail\_emp}}{\partial r} + z_{tr}\frac{\partial T_{tr}}{\partial r}$$

For M_w, M_{B1}, we have:

$$\frac{\partial M}{\partial w} = M_{MR}\frac{\partial a_{1S}}{\partial w} + h_z T_{TPP}\frac{\partial a_{1s}}{\partial w} + h_z a_{1S}\frac{\partial T_{TPP}}{\partial w} + h_x\frac{\partial T_{TPP}}{\partial w} + h_x\frac{\partial H_{TPP}}{\partial w} + \frac{\partial M_{F+t\_e}}{\partial w}$$

$$\frac{\partial M}{\partial B_1} = M_{MR}\frac{\partial a_{1S}}{\partial B_1} + h_z T_{TPP}\frac{\partial a_{1s}}{\partial B_1} + h_z a_{1S}\frac{\partial T_{TPP}}{\partial B_1} + h_x\frac{\partial T_{TPP}}{\partial B_1} + h_x\frac{\partial H_{TPP}}{\partial B_1} + \frac{\partial M_{F+t\_e}}{\partial B_1}$$

For N_r, $N_{\theta tr}$ we have:

$$\frac{\partial N}{\partial r} = N_r = -T_{TPP} h_x\frac{\partial b_{1S}}{\partial r} - h_x b_{1S}\frac{\partial T_{TPP}}{\partial r} + \frac{\partial N_{Fus+tail\_emp}}{\partial r} + l_{tr}\frac{\partial T_{tr}}{\partial r} + \frac{\partial Q}{\partial r}$$

$$\frac{\partial N}{\partial \theta_{tr}} = N_{\theta_{tr}} = -T_{TPP} h_x \frac{\partial b_{1S}}{\partial \theta_{tr}} - h_x b_{1S} \frac{\partial T_{TPP}}{\partial \theta_{tr}} + \frac{\partial N_{Fus+tail\_emp}}{\partial \theta_{tr}} + l_{tr} \frac{\partial T_{tr}}{\partial \theta_{tr}} + \frac{\partial Q}{\partial \theta_{tr}}$$

7.5.3. *Notes on the methodology of small perturbations*

From the set of Equations (7.4) the reader may appreciate that it is not possible, from a rigorous standpoint, to separate a pure lateral-directional motion. Consequently, the six equations require to be solved simultaneously.

The small disturbance assumption has reduced the interaction between the longitudinal motion and the lateral motion. Moreover, considering the groups of Equations (7.6) and (7.7), it is more clear the link (by means of the variables involved) among the first, the third and the fifth equations (they define the longitudinal motion), and among the second, the fourth and the sixth equations (they define the lateral-directional motion).

Therefore, in order to simplify the treatment, the two types of motions will be analyzed separately, knowing both the approximations made. In the developments that follow, for the purposes of this chapter (which provides an introduction to stability and control), we will study separately the longitudinal motion and the lateral-directional motion. From a mathematical point of view, the problem requires to determine the equations by introducing the derivatives, and then to perform the stability analysis of the equations obtained.

7.6. Dynamic stability in hovering flight

7.6.1. *Longitudinal dynamic stability in hovering flight*

Considering Equations (7.7), we start with the longitudinal motion by using the expressions with the derivatives; however, we do not take into account those stability derivatives that we have neglected by applying the separation of the lateral-directional motion from the longitudinal motion. Therefore, we obtain:

$$\begin{cases} \dfrac{W_G}{g}(\dot{u}) = X - W_G[\Theta_d] \\ \dfrac{W_G}{g}(\dot{w}) = Z \\ \dot{q} I_y = M \end{cases}$$

and, finally

$$\begin{cases} \dfrac{W_G}{g}(\dot{u}) = X_u u + X_w w + X_q q + X_{\theta_{MR}}\theta_{MR} + X_{B_1}B_1 - W_G[\Theta_d] \\ \dfrac{W_G}{g}(\dot{w}) = Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + Z_{\theta_{MR}}\theta_{MR} + Z_{B_1}B_1 \\ \dot{q}I_y = M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\theta_{MR}}\theta_{MR} + M_{B_1}B_1 \end{cases} \tag{7.8}$$

Then, we rewrite the right-hand side where we insert only the control terms; therefore, we obtain:

$$\begin{cases} -\dfrac{W_G}{g}(\dot{u}) - W_G[\Theta_d] + X_u u + X_w w + X_q q = -X_{\theta_{MR}}\theta_{MR} - X_{B_1}B_1 \\ -\dfrac{W_G}{g}(\dot{w}) + Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q = -Z_{\theta_{MR}}\theta_{MR} - Z_{B_1}B_1 \\ -\dot{q}I_y + M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q = -M_{\theta_{MR}}\theta_{MR} - M_{B_1}B_1 \end{cases} \tag{7.9}$$

Before proceeding with the analysis of Equations (7.9), it is important to remember the meaning of some stability derivatives. This will also help the interpretation of other derivatives. In previous analysis of static stability we saw that an increment u in forward speed along the X-axis produces an increase and a decrease in airspeed at the advancing blade and at the retreating blade, respectively. As final result, the main rotor tilts backwards, with an increase in thrust, in *H*-force and in longitudinal aerodynamic drag of the fuselage. Then, it is clear that these changes are functions of the forward flight speed of the helicopter (higher the flight speed, higher the changes). Instead, in hover, a small disturbance u does not involve meaningful changes for the force along the Z-axis: then, $\partial Z/\partial u$ can be neglected. Otherwise, as forward speed increases, this assumption becomes not acceptable, because at low speeds $\partial Z/\partial u < 0$, and then, at high speeds $\partial Z/\partial u > 0$. Moreover, an increment w in speed along the Z-axis will produce an effect on the forces along the Z-axis, but will not produce a significant effect along the X-axis (especially in hovering flight); then, in this flight condition we can accept that $\partial X/\partial w = 0$. Analogously, we can verify that in hovering is acceptable to consider $\partial Z/\partial q = 0$, $\partial Z/\partial \dot{w} = 0$ and $\partial M/\partial \dot{w} = 0$.

Using the previous results, the group of Equations (7.9) may be simplified as:

$$\begin{cases} -\dfrac{W_G}{g}(\dot{u}) - W_G[\Theta_d] + X_u u + X_q q = -X_{\theta_{MR}}\theta_{MR} - X_{B_1}B_1 \\ -\dfrac{W_G}{g}(\dot{w}) + Z_w w = -Z_{\theta_{MR}}\theta_{MR} - Z_{B_1}B_1 \\ -\dot{q}I_y + M_u u + M_w w + M_q q = -M_{\theta_{MR}}\theta_{MR} - M_{B_1}B_1 \end{cases} \tag{7.10}$$

This set of equations is composed of linear differential equations with constant coefficients. Now, the objective is to investigate about the typology of stability following a disturbance.

From the theory of differential equations, it is known that some tools for the immediate stability verification are available. Remember Routh's criterion for stability that allows us to proceed without the necessity to solve the equations involved. However, the criterion (non-quantitative type) presents operational limits because it does not allow us to evaluate the entity of stability or instability of the system.

7.6.1.1. *Equations of motion, state variable form*

Let us divide the derivatives by the mass M_{heli} of the helicopter or by the moment of inertia I_y, as follows:

$$X_u^\circ = \frac{X_u}{M_{heli}}; Z_w^\circ = \frac{Z_w}{M_{heli}}; M_u^\circ = \frac{M_u}{I_y}; M_w^\circ = \frac{M_w}{I_y}; M_q^\circ = \frac{M_q}{I_y}; \tag{7.11}$$

By using the same procedure we obtain $X^o_{\theta_{MR}}$, $X^o_{B_1}$, $Z^0_{\theta_{MR}}$, $Z^0_{B_1}$, $M^0_{\theta_{MR}}$, $M^0_{B_1}$.

By using expressions (7.11), the set of linear differential equations with constant coefficients (7.10) can be written in a more compact manner, introducing *the state vector* **x**.

Then, the set of equations may be written in matrix form as

$$\dot{\mathbf{x}} = \mathbf{A}_d\mathbf{x} + \mathbf{B}\mathbf{c}$$

where

- **x** the state vector
- $\mathbf{A}_d$ the stability derivatives matrix
- **B** the control matrix
- **c** the control vector

and also:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\Theta}_d \end{bmatrix} = \begin{bmatrix} X_u^0 & 0 & X_q^0 & -g \\ 0 & Z_w^0 & 0 & 0 \\ M_u^0 & M_w^0 & M_q^0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \Theta_d \end{bmatrix} + \begin{bmatrix} X_{\theta_{MR}}^o & X_{B_1}^o \\ Z_{\theta_{MR}}^0 & Z_{B_1}^0 \\ M_{\theta_{MR}}^0 & M_{B_1}^0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{MR} \\ B_1 \end{bmatrix}$$

Now, we assume that $\theta_{MR} = 0$, $B_1 = 0$. The characteristic polynomial $\varphi(\lambda)$ is equal to det(λ **I** - $\mathbf{A}_d$). Therefore the characteristic equation is obtained expanding the following determinant:

$$\varphi(\lambda) = \det(\lambda\, \mathbf{I} - \mathbf{A}_d) = 0$$

where **I** is the identity matrix, order 4x4.
Being

$$\lambda \mathbf{I} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

then, we have

$$\begin{vmatrix} \lambda - X_u^\circ & 0 & -X_q^0 & g \\ 0 & \lambda - Z_w^0 & 0 & 0 \\ -M_u^0 & -M_w^0 & \lambda - M_q^0 & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = 0 \qquad (7.12a)$$

Expanding the determinant produces the following characteristic equation:

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \qquad (7.12b)$$

where

$$A = 1$$
$$B = -X_u^0 - Z_w^0 - M_q^0$$

$$C = Z_w^0(X_u^0 + M_q^0)$$
$$D = M_u^0 g$$
$$E = -Z_w^0 M_u^0 g$$

The expressions for the terms *C* and *D* are obtained considering that

$$[\,C = Z_w^0(X_u^0 + M_q^0) + X_u^0 M_q^0 - M_u^0 X_q^0\,]$$
$$[\,D = M_u^0 g - Z_w^0(X_u^0 M_q^0 - M_u^0 X_q^0)\,]$$

and, for the helicopter with a single main rotor, note that:

$$[\,X_u^0 M_q^0 - M_u^0 X_q^0 = 0\,]$$

Now, in order to solve the characteristic Equation (7.12b), first of all the values of the stability derivatives involved shall be calculated.

Then, the characteristic Equation (7.12b) has four roots: λ_1, λ_2, λ_3, λ_4, eigenvalues of matrix $\mathbf{A}_d$, and they may be real or complex conjugate. Therefore, the generic root λ has the form $\lambda = \eta \pm i\omega$.

The general solution for each dependent variable (for example, we choose *u*) is of the following type:

$$u = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t}$$

where a_1, a_2, a_3, a_4 are constant that can be evaluated by the initial conditions. Consequently, for stability verification tasks, if the real roots are negative, then the perturbation is damped; vice versa if the real roots are positive, then the perturbed motion results a divergence. In case of complex roots, if the real part is negative, then the motion is a damped oscillation; vice versa, if the real part is positive, then the motion is a divergent oscillation (Figure 7.5).

In hover, the typical case for a single main rotor helicopter is dominated by a couple of real roots and a couple of complex conjugate roots.

Real roots are related to very damped motions (*heavily damped subsidence*) with a pure convergence, while the complex roots configure a dynamically unstable motion, with increasing amplitude oscillations.

To understand the unstable response it is necessary to remember, for example, the response of rotor to a forward speed disturbance:

the rotor tilts backwards, producing a nose up attitude of the helicopter. Then, a backward motion is generated and, now, the rotor tilts forwards causing a nose down attitude of the helicopter. We immediately understand that the backward motion is stopped, but a forward motion is starting, so the phenomenon restarts in a manner that is divergent and unstable.
In this case, the meaningful stability derivative is $\partial M/\partial q$, and the designer shall consider in detail this derivative in order to attenuate the unstable motion.

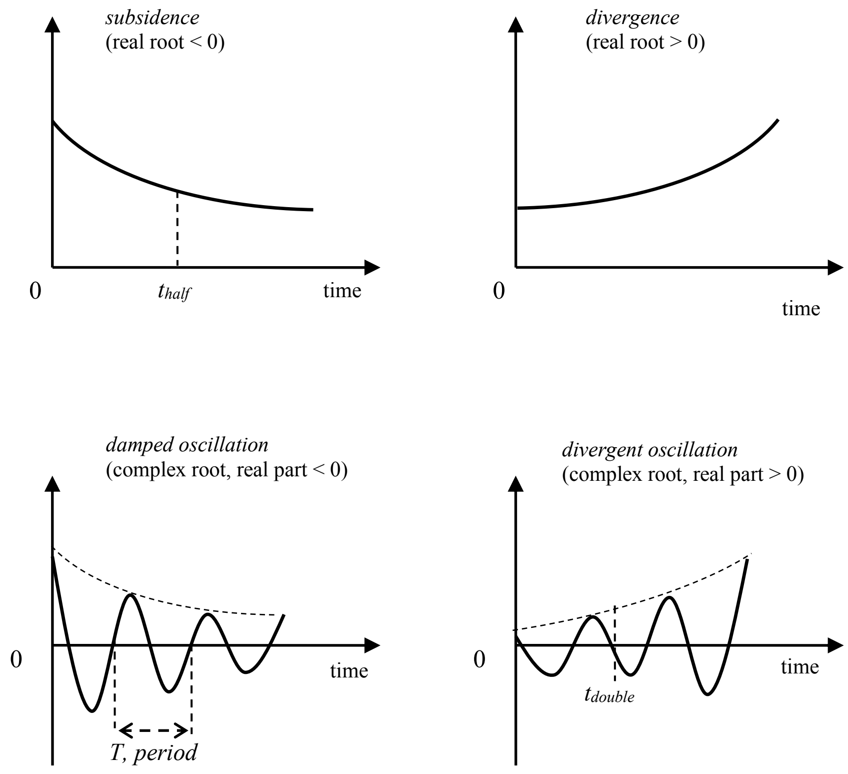

Figure 7.5 Evolutions of perturbed motion

Generally, for calculation of the oscillation frequency, the second equation of the set of Equations (7.10) is considered negligible, because the described motion does not involve relevant changes in altitude. Consequently, considering the first equation and the third equation, we can write the characteristic equation that provides the *oscillation frequency*:

$$\omega = \sqrt{\frac{-gM_u^0}{M_q^0}}$$

7.6.1.2. *The stability derivatives M_q and M_u in hover*

In hover, the only contribution that cannot be neglected for the calculation of the derivative $\partial M/\partial q$ is produced by the main rotor. Therefore, considering the parameters in Figure 5.2b (Chapter 5), we have:

$$M_q = (M_q)_{MR} = \frac{\partial M}{\partial q} = \frac{\partial Z}{\partial q} \cdot h_x - \frac{\partial X}{\partial q} \cdot h_z \qquad \text{(for rotor with hinge offset } \varepsilon_\beta \approx 0\text{)}$$

or

$$M_q = (M_q)_{MR} = \frac{\partial M}{\partial q} = \frac{\partial Z}{\partial q} \cdot h_x - \frac{\partial X}{\partial q} \cdot h_z + \frac{dM_{MR}}{da_{1S}} \frac{\partial a_{1S}}{\partial q} \qquad \text{(for hingeless rotor)}$$

These expressions may be simplified remembering that $\partial Z/\partial q \approx 0$ in hover.
In a similar manner, we have:

$$M_u = (M_u)_{MR} = \frac{\partial M}{\partial u} = -\frac{\partial X}{\partial u} \cdot h_z \qquad \text{(for rotor with hinge offset } \varepsilon_\beta \approx 0\text{)}$$

or

$$M_u = (M_u)_{MR} = \frac{\partial M}{\partial u} = -\frac{\partial X}{\partial u} \cdot h_z + \frac{dM_{MR}}{da_{1S}} \frac{\partial a_{1S}}{\partial u} \qquad \text{(for hingeless rotor)}$$

considering that, in hovering flight, $\partial Z/\partial u \approx 0$.
To estimate the previous terms, we have to consider that

$$X_u = (X_u)_{MR} = -\frac{\partial H_{TPP}}{\partial u};$$

$$X_q = (X_q)_{MR} = -\frac{\partial H_{TPP}}{\partial q}$$

7.6.1.3 *Approximate calculation of longitudinal modes in hovering flight for a medium helicopter*

In order to illustrate the previous theory, we will study the approximate control fixed response in hovering flight at sea level of a typical medium helicopter (DL= 350 N/m^2; main rotor: four-bladed rotor, radius R = 6.6 m, chord c = 0.4 m; tail rotor: radius R_{tr} = 1.0 m; l_{tr} = 8.1 m). Now, let us consider that the applicable stability derivatives assume the values shown in the matrix $\mathbf{A}_d$ as follows:

$$\mathbf{A}_d = \begin{bmatrix} -0.0200 & 0 & 0.8500 & -9.8066 \\ 0 & -0.300 & 0 & 0 \\ 0.0500 & 0.065 & -1.700 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, the characteristic equation (7.12b) becomes:

$$\lambda^4 + (2.0200)\lambda^3 + (0.516)\lambda^2 + (0.4903)\lambda + (0.1471) = 0$$

The roots of the characteristic equation above, or eigenvalues, are:

$$\lambda_1 = -0.300, \qquad \lambda_2 = -1.861, \qquad \lambda_{3,4} = 0.0707 \pm 0.5083i$$

Therefore, the two negative real roots represent two stable responses with a t_{half} (time to half, or time during which the disturbance quantity will half itself) equal to:

$$t_{half} = \frac{0.693}{|n|}$$

and:

$$t_{half} = \frac{0.693}{|n|} = \frac{0.693}{|-0.300|} = 2.31\,\text{seconds} \quad (\text{for } \lambda_1),$$

$$t_{half} = \frac{0.693}{|n|} = \frac{0.693}{|-1.861|} = 0.37\,\text{seconds} \quad (\text{for } \lambda_2)$$

The complex roots λ_3 and λ_4 (that have a positive real parts) imply an unstable oscillatory mode (divergent oscillation) with the following period T, time to double amplitude t_{double}, undamped natural frequency ω_n, damping ratio ζ, and number of cycles to double amplitude N_{double}:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.5083} = 12.4 \text{ seconds}, \qquad t_{double} = \frac{0.693}{|n|} = \frac{0.693}{0.0707} = 9.8 \text{ seconds}$$

$$\omega_n = \sqrt{n^2 + \omega^2} = 0.513 \text{ rad/s}, \qquad \zeta = \frac{-n}{\omega_n} = \frac{-0.0707}{0.513} = 0.138,$$

$$N_{double} = \frac{0.693}{2\pi} \frac{\sqrt{1-\zeta^2}}{|\zeta|} = 0.7895$$

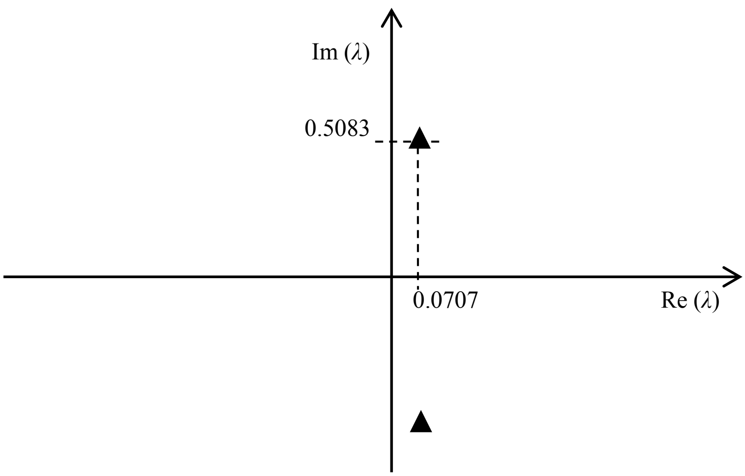

Figure 7.6 Roots λ_3 and λ_4 on complex plane

7.6.1.4. *The characteristic roots on complex plane*

In the preceding example of calculation we obtained the numerical parameters related to each root, real or complex. From a general standpoint, now it is useful to show the relationships among n, ω_n and ζ in the complex plane.

Figure 7.7 shows a generic root $\lambda = n \pm i\omega$ in the left half plane (therefore, n is negative):

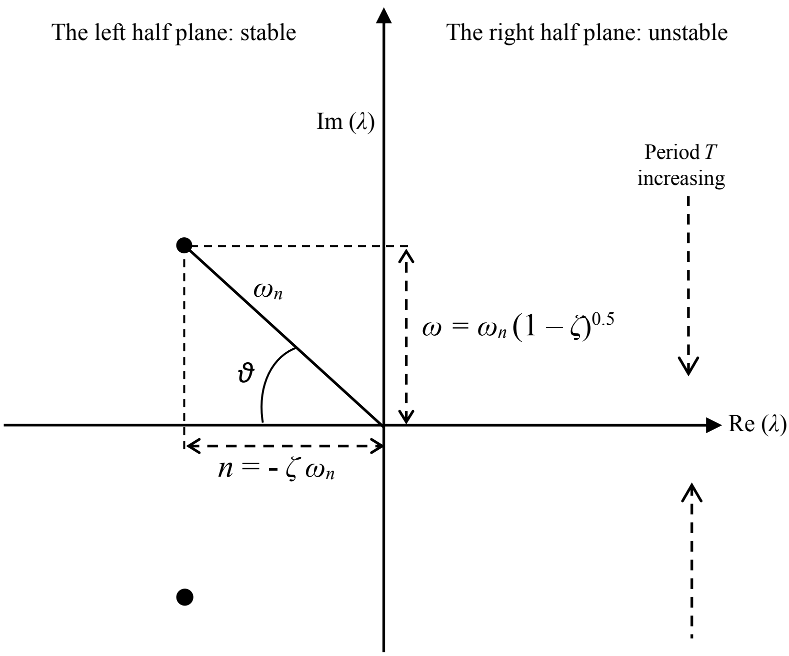

Figure 7.7 Relationships among parameters on complex plane

7.6.2. *Lateral-directional dynamic stability in hovering flight*

Now, for the lateral-directional motion, we saw that the applicable set of equations is:

$$\begin{cases} \dfrac{W_G}{g}(\dot{v}) = Y + W_G[\Phi_d] \\ \dot{p}I_x - \dot{r}I_{xz} = L \\ \dot{r}I_z - \dot{p}I_{xz} = N \end{cases}$$

By substituting the expressions of perturbations into equations above, we obtain:

$$\begin{cases} -\dfrac{W_G}{g}(\dot{v}) + W_G[\Phi_d] + Y_v v + Y_p p + Y_r r + Y_{A_1} A_1 = -Y_{\theta_{tr}} \theta_{tr} \\ \dot{p} I_x - \dot{r} I_{xz} = L_v v + L_p p + L_r r + L_{A_1} A_1 + L_{\theta_{tr}} \theta_{tr} \\ \dot{r} I_z - \dot{p} I_{xz} = N_v v + N_p p + N_r r + N_{A_1} A_1 + N_{\theta_{tr}} \theta_{tr} \end{cases} \quad (7.13)$$

Now, let us divide the stability derivatives of the first equation by the mass M_{heli} of the helicopter; thus, we obtain:

$$Y_p^0 = \frac{Y_p}{M_{heli}}, \quad Y_r^0 = \frac{Y_r}{M_{heli}}, \quad Y_v^0 = \frac{Y_v}{M_{heli}}, \quad Y_{A_1}^0 = \frac{Y_{A_1}}{M_{heli}}, \quad Y_{\theta_{tr}}^0 = \frac{Y_{\theta_{tr}}}{M_{heli}} \quad (7.14a)$$

Considering the second and the third equations of the set (7.13), in order to write the group of equations in the required matrix form, let us calculate the expression of the term $\dot{r}I_{xz}$ from the third equation; we have:

$$I_{xz}\dot{r} = \frac{I_{xz}^2}{I_z}\dot{p} + \frac{I_{xz}}{I_z}N_v v + \frac{I_{xz}}{I_z}N_p p + \frac{I_{xz}}{I_z}N_r r + \frac{I_{xz}}{I_z}N_{A_1}A_1 + \frac{I_{xz}}{I_z}N_{\theta_{tr}}\theta_{tr}$$

By substituting into the second equation, we obtain:

$$\dot{p}(I_x - \frac{I_{xz}^2}{I_z}) = \frac{I_{xz}}{I_z}N_v v + \frac{I_{xz}}{I_z}N_p p + \frac{I_{xz}}{I_z}N_r r + \frac{I_{xz}}{I_z}N_{A_1}A_1$$
$$+ \frac{I_{xz}}{I_z}N_{\theta_{tr}}\theta_{tr} + L_v v + L_p p + L_r r + L_{A_1}A_1 + L_{\theta_{tr}}\theta_{tr}$$

Multiplying by I_z and after appropriate rearranging of terms, finally, is:

$$\dot{p} = \frac{(I_{xz}N_v + I_z L_v)}{(I_x I_z - I_{xz}^2)} v + \frac{(I_{xz}N_p + I_z L_p)}{(I_x I_z - I_{xz}^2)} p + \frac{(I_{xz}N_r + I_z L_r)}{(I_x I_z - I_{xz}^2)} r$$
$$+ \frac{(I_{xz}N_{A_1} + I_z L_{A_{11}})}{(I_x I_z - I_{xz}^2)} A_1 + \frac{(I_{xz}N_{\theta_{tr}} + I_z L_{\theta_{tr}})}{(I_x I_z - I_{xz}^2)} \theta_{tr}$$

By applying a similar procedure to the third equation, we can calculate the expression for $\dot{r}$.

Therefore, in order to write the previous expressions in a suitable form, let us use the following relations:

$$L_p^0 = \frac{I_z(L_p) + I_{xz}(N_p)}{I_x I_z - I_{xz}^2} \; ; \qquad N_p^0 = \frac{I_x(N_p) + I_{xz}(L_p)}{I_x I_z - I_{xz}^2}$$

$$L_r^0 = \frac{I_z(L_r) + I_{xz}(N_r)}{I_x I_z - I_{xz}^2} \; ; \qquad N_r^0 = \frac{I_x(N_r) + I_{xz}(L_r)}{I_x I_z - I_{xz}^2}$$

$$L_v^0 = \frac{I_z(L_v) + I_{xz}(N_v)}{I_x I_z - I_{xz}^2} \; ; \qquad N_v^0 = \frac{I_x(N_v) + I_{xz}(L_v)}{I_x I_z - I_{xz}^2} \qquad (7.14b)$$

$$L_{A_1}^0 = \frac{I_z(L_{A_1}) + I_{xz}(N_{A_1})}{I_x I_z - I_{xz}^2} \; ; \qquad N_p^0 = \frac{I_x(N_{A_1}) + I_{xz}(L_{A_1})}{I_x I_z - I_{xz}^2}$$

$$L_{\theta_{tr}}^0 = \frac{I_z(L_{\theta_{tr}}) + I_{xz}(N_{\theta_{tr}})}{I_x I_z - I_{xz}^2} \; ; \qquad N_{\theta_{tr}}^0 = \frac{I_x(N_{\theta_{tr}}) + I_{xz}(L_{\theta_{tr}})}{I_x I_z - I_{xz}^2}$$

As in the longitudinal motion we just treated, let us rewrite the group of Equations (7.13) by expressions (7.14a) and (7.14b); then using the matrix notation, we have:

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\Phi}_d \\ \dot{\Psi}_d \end{bmatrix} = \begin{bmatrix} Y_v^0 & Y_p^0 & Y_r^0 & g & 0 \\ L_v^0 & L_p^0 & L_r^0 & 0 & 0 \\ N_v^0 & N_p^0 & N_r^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \Phi_d \\ \Psi_d \end{bmatrix} + \begin{bmatrix} Y_{A_1}^0 & Y_{\theta_{tr}}^0 \\ L_{A_1}^0 & L_{\theta_{tr}}^0 \\ N_{A_1}^0 & N_{\theta_{tr}}^0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ \theta_{tr} \end{bmatrix}$$

Expanding the following determinant

$$\begin{vmatrix} \lambda - Y_v^0 & -Y_p^0 & -Y_r^0 & -g & 0 \\ -L_v^0 & \lambda - L_p^0 & -L_r^0 & 0 & 0 \\ -N_v^0 & -N_p^0 & \lambda - N_r^0 & 0 & 0 \\ 0 & -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & \lambda \end{vmatrix} = 0$$

leads to the characteristic equation:

$$A\lambda^5 + B\lambda^4 + C\lambda^3 + D\lambda^2 + E\lambda + F = 0$$

where

$$A = 1$$
$$B = -N_r^0 - L_p^0 - Y_v^0$$
$$C = L_p^0 N_r^0 - N_p^0 L_r^0 + Y_v^o N_r^0 + Y_v^0 L_p^0 - L_v^0 Y_p^0 - N_v^0 Y_r^0$$
$$D = Y_v^o (L_r^o N_p^0 - L_p^0 N_r^0) + L_v^0 (Y_p^0 N_r^0 - N_p^0 Y_r^0 - g) + N_v (Y_r^0 L_p^0 - Y_p^0 L_r^0)$$
$$E = L_v^0 N_r^0 g - L_r^0 N_v^0 g$$
$$F = 0$$

In this case, there are five roots; in detail, one is relative to $\lambda = 0$ and represents a neutral condition of stability (*heading mode*); other four roots, typically, are as follows:

— two real roots, relative to stable motions (one root produces a rolling damped motion, the other one produces a yawing damped motion);
— two complex conjugate roots, that produce dynamically unstable oscillation.

The rolling damped mode is characterized by the derivative L_p, while the yaw stable mode by the derivative N_r.
The unstable oscillation represents changes in helicopter sideways speed and in bank angle.

7.7. Dynamic Stability in forward flight

7.7.1. *Longitudinal dynamic stability in forward flight*

In this flight condition, the set of equations is:

$$\begin{cases} \dfrac{W_G}{g}(\dot{u} + V_{z0} q) = X - W_G [\Theta_d \cos\Theta_0] \\ \dfrac{W_G}{g}(\dot{w} - V_{x0} q) = Z - W_G (\Theta_d \sin\Theta_0) \\ \dot{q} I_y = M \end{cases}$$

Introducing the stability derivatives, the equations become:

$$\begin{cases} -\dfrac{W_G}{g}(\dot{u})-W_G[\Theta_d\cos\Theta_0]+X_u u+X_w w+(X_q-\dfrac{W_G}{g}V_{z0})q=-X_{\theta_{MR}}\theta_{MR}-X_{B_1}B_1 \\ -\dfrac{W_G}{g}(\dot{w})-W_G(\Theta_d sen\Theta_0)+Z_u u+Z_w w+(Z_q+\dfrac{W_G}{g}V_{x0})q=-Z_{\theta_{MR}}\theta_{MR}-Z_{B_1}B_1 \\ -\dot{q}I_y+M_u u+M_w w+M_{\dot{w}}\dot{w}+M_q q=-M_{\theta_{MR}}\theta_{MR}-M_{B_1}B_1 \end{cases}$$

Also in this case, the derivative $\partial M/\partial\dot{w}$ can be neglected to simplify the calculation and, therefore, can be removed from the third equation.
Using the expressions (7.11), the above set of equations may be written in matrix notation (using a similar procedure just applied to hovering flight condition):

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\Theta}_d \end{bmatrix} = \begin{bmatrix} X_u^0 & X_w^0 & X_q^0-V_{z0} & -g\cos\Theta_0 \\ Z_u^0 & Z_w^0 & Z_q^0+V_{x0} & -g\sin\Theta_0 \\ M_u^0 & M_w^0 & M_q^0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \Theta_d \end{bmatrix} + \begin{bmatrix} X_{\theta_{MR}}^o & X_{B_1}^o \\ Z_{\theta_{MR}}^0 & Z_{B_1}^0 \\ M_{\theta_{MR}}^0 & M_{B_1}^0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{MR} \\ B_1 \end{bmatrix}$$

Now, we study the fixed control response (natural modes of motion) of the helicopter; therefore, we need to expand the following determinant:

$$\begin{vmatrix} \lambda-X_u^0 & -X_w^0 & -X_q^0+V_{z0} & g\cos\Theta_0 \\ -Z_u^0 & \lambda-Z_w^0 & -Z_q^0-V_{x0} & g\sin\Theta_0 \\ -M_u^0 & -M_w^0 & \lambda-M_q^0 & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix}=0$$

that leads finally to the characteristic equation

$$A\lambda^4+B\lambda^3+C\lambda^2+D\lambda+E=0 \tag{7.15}$$

where

$$A=1$$
$$B=-M_q^0-Z_w^0-X_u^0$$
$$C=-M_u^0X_q^0+Z_w^0M_q^0-M_w^0Z_q^0-M_w^0V_{x0}+X_u^0M_q^0+X_u^0Z_w^0-Z_u^0X_w^0$$

$$+M_u^0 V_{z0}$$

$$D = M_u^0 g\cos\Theta_0 + M_w^0 g\sin\Theta_0 - M_u^0 X_w^0 V_{x0} + M_u^0 Z_w^0 X_q^0 - M_u^0 Z_w^0 V_{z0} + Z_u^0 M_w^0 V_{z0}$$

$$- Z_u^0 M_w^0 X_q^0 - M_u^0 X_w^0 Z_q^0 - X_u^o Z_w^0 M_q^0 + X_u^0 M_w^0 Z_q^0 + X_u^0 M_w^0 V_{x0} + Z_u^0 X_w^0 M_q^0$$

$$E = (M_u^0 X_w^0 - X_u^0 M_w^0) g\sin\Theta_0 + (Z_u^0 M_w^0 - M_u^0 Z_w^0) g\cos\Theta_0$$

Before analyzing the roots of the characteristic equation, it shall be noted that, generally, the values of the stability derivatives can vary throughout the flight envelope of the helicopter, from hovering to high-speed forward flight. Consequently, the trend of the stability derivatives has an impact on the typology of the roots so that the characteristic equation could have two pairs of complex conjugate roots or four real roots.

In the case of two pairs of complex conjugate roots, a response similar to that of fixed-wing aircraft is obtained, with a pair of complex roots that corresponds to an oscillatory motion, with a long period, defined as *phogoid mode* (Figure 7.8).

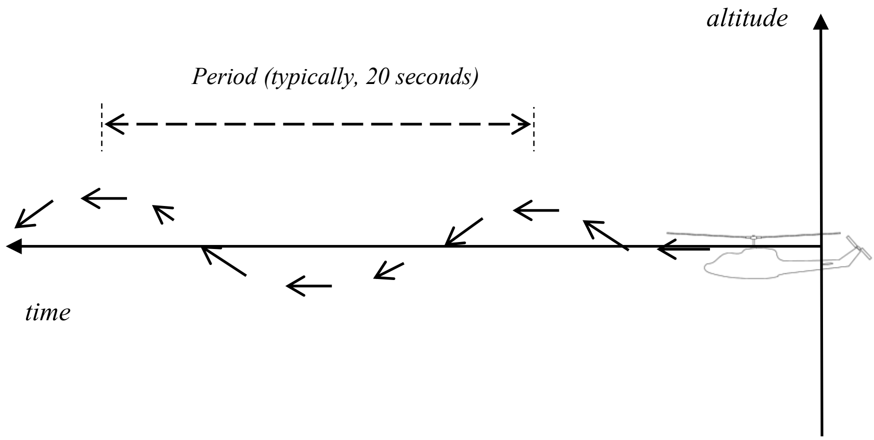

Figure 7.8 Phogoid mode

The long period oscillatory motion, the phugoid mode, is characterized by changes in altitude and speed, with an angle of attack almost constant.

In order to understand the motion represented, let us assume that, following a disturbance, the altitude increases: now, during the climb, a decrease in speed and the action produced by the weight take the helicopter to descend. Decreasing the altitude, the speed and the rotor thrust increase so that the oscillation restarts once again, and generally the motion is unstable. For this case, the very relevant stability derivatives are $M_w^o, M_q^0, M_u^0, Z_u^0$.

7.7.1.1. *Approximate calculation of longitudinal modes in forward level flight for a medium helicopter*

For example, we will study the approximate control fixed response of a utility helicopter (see also calculation in Section 7.6.1.3) in straight and level flight at *V*=100 knots, sea level, where we assume that the characteristic equation (7.15) becomes:

$$\lambda^4 + (3.3400)\lambda^3 + (0.4333)\lambda^2 + (0.2205)\lambda + (0.2414) = 0$$

The roots of the characteristic equation above, or eigenvalues, are:

$$\lambda_1 = -0.4266\,, \qquad \lambda_2 = -3.2195, \qquad \lambda_{3,4} = 0.1530 \pm 0.3903i$$

Therefore, in this case we have two negative real roots which represent two stable responses with a t_{half} equal to:

$$t_{half} = \frac{0.693}{|n|}$$

and:

$$t_{half} = \frac{0.693}{|n|} = \frac{0.693}{|-0.4266|} = 1.6 \text{ seconds} \qquad (\text{for } \lambda_1),$$

$$t_{half} = \frac{0.693}{|n|} = \frac{0.693}{|-3.2195|} = 0.2 \text{ seconds} \qquad (\text{for } \lambda_2)$$

From results above, we see that the stable modes are short-period responses.

The complex roots λ_3 and λ_4 (that have a positive real parts) imply an unstable mode (the phugoid) with the following period T, time to double amplitude t_{double}, undamped natural frequency ω_n and damping ratio ζ:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.3903} = 16.1 \text{ seconds},$$

$$t_{double} = \frac{0.693}{|n|} = \frac{0.693}{0.1530} = 4.5 \text{ seconds}$$

$$\omega_n = \sqrt{n^2 + \omega^2} = 0.419 \text{ rad/s},$$

$$\zeta = \frac{-n}{\omega_n} = \frac{-0.1530}{0.419} = 0.365$$

$$N_{double} = \frac{0.693}{2\pi} \frac{\sqrt{1-\zeta^2}}{|\zeta|} = 0.2806$$

Example of longitudinal root locus plot as a function of forward flight speed
In Sections 7.6.1.3 and 7.7.1.1 we studied the longitudinal natural modes, in hovering flight and at V= 100 knots respectively, of a medium reference helicopter with a hingeless rotor, where we assumed also uncoupled longitudinal and lateral-directional motions. Again, we remember that this a critical assumption for the helicopter, because it is characterized by an asymmetric configuration. The fully coupled equations can show significant different results both in longitudinal and lateral-directional eigenvalues with respect to results provided by the analysis of the uncoupled set, from the hovering to the forward flight. This must be always considered when a rigorous analysis shall be performed.
In particular, the following derivatives, which are neglected in the uncoupled analysis, shall be considered:
L_u (roll moment due to longitudinal velocity), L_w (roll moment due to vertical velocity), L_q (roll moment due to pitch rate), M_v (pitch moment due to the lateral velocity), M_p (pitch moment due to the roll rate), N_w (yaw moment due to the vertical velocity), and the other control derivatives.

Figure 7.9 illustrates the influence of the forward speed, from hovering to forward flight at V=100 knots at sea level, on the longitudinal eigenvalues for the helicopter used for calculation.

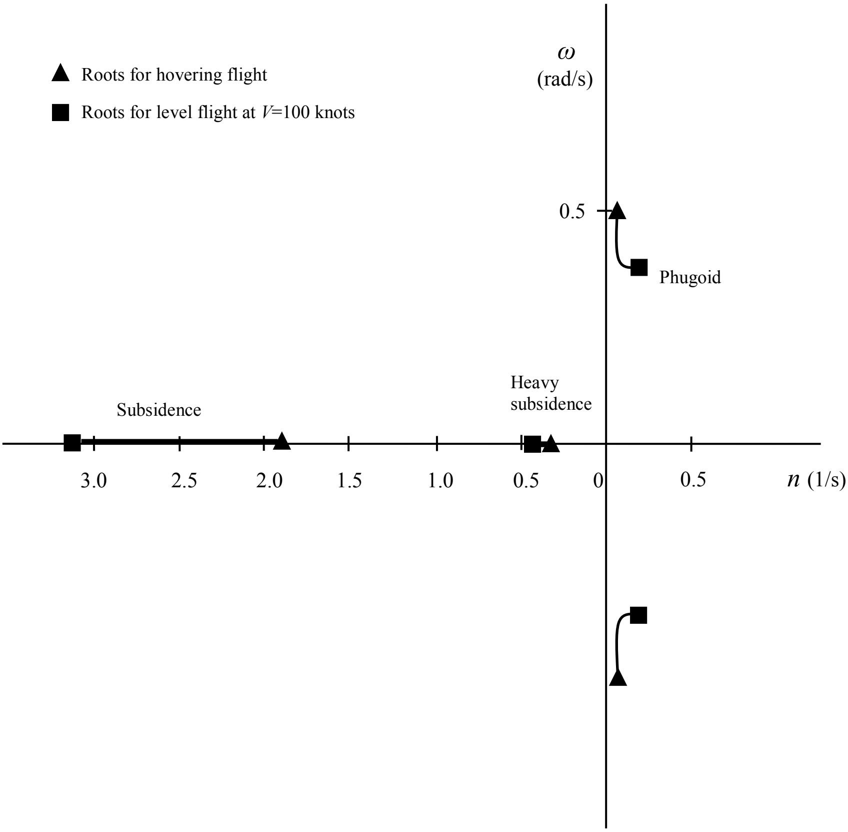

Figure 7.9 Example of longitudinal root locus as a function of forward speed

7.7.2. *Lateral-directional dynamic stability in forward flight*

In a similar manner, the set of equations for lateral-directional flight condition is:

$$\begin{cases} \dfrac{W_G}{g}(\dot{v} + V_{x0}r - V_{z0}p) = Y + W_G[\Phi_d \cos\Theta_0] \\ \dot{p}I_x - \dot{r}I_{xz} = L \\ \dot{r}I_z - \dot{p}I_{xz} = N \end{cases}$$

By substituting the expressions for perturbations, the equations above become:

$$\begin{cases} \frac{W_G}{g}(\dot{v}+V_{x0}r-V_{z0}p)=W_G[\Phi_d\cos\Theta_0]+Y_v v+Y_p p+Y_r r+Y_{A_1}A_1+Y_{\theta_{tr}}\theta_{tr} \\ \dot{p}I_x-\dot{r}I_{xz}=L_v v+L_p p+L_r r+L_{A_1}A_1+L_{\theta_{tr}}\theta_{tr} \\ \dot{r}I_z-\dot{p}I_{xz}=N_v v+N_p p+N_r r+N_{A_1}A_1+N_{\theta_{tr}}\theta_{tr} \end{cases}$$

and then:

$$\begin{cases} -\frac{W_G}{g}(\dot{v}+V_{x0}r-V_{z0}p)+W_G[\Phi_d\cos\Theta_0]+Y_v v+Y_p p+Y_r r=-Y_{A_1}A_1-Y_{\theta_{tr}}\theta_{tr} \\ -\dot{p}I_x+\dot{r}I_{xz}+L_v v+L_p p+L_r r=-L_{A_1}A_1-L_{\theta_{tr}}\theta_{tr} \\ -\dot{r}I_z+\dot{p}I_{xz}+N_v v+N_p p+N_r r=-N_{A_1}A_1-N_{\theta_{tr}}\theta_{tr} \end{cases}$$

By using expressions (7.14a) and (7.14b) and by means of the procedure applied to the previous cases, we can rewrite the equations above, and the matrix $\mathbf{A}_d$ becomes:

$$\mathbf{A}_d=\begin{bmatrix} Y_v^0 & Y_p^0+V_{z0} & -V_{x0}+Y_r^0 & g\cos\Theta_0 & 0 \\ L_v^0 & L_p^0 & L_r^0 & 0 & 0 \\ N_v^0 & N_p^0 & N_r^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore, we obtain the following determinant:

$$\begin{vmatrix} \lambda-Y_v^0 & -Y_p^0-V_{z0} & V_{x0}-Y_r^0 & -g\cos\Theta_0 & 0 \\ -L_v^0 & \lambda-L_p^0 & -L_r^0 & 0 & 0 \\ -N_v^0 & -N_p^0 & \lambda-N_r^0 & 0 & 0 \\ 0 & -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & \lambda \end{vmatrix}=0$$

and finally, we have the characteristic equation

$$A\lambda^5 + B\lambda^4 + C\lambda^3 + D\lambda^2 + E\lambda + F = 0$$

where:

$$A = 1$$
$$B = -Y_v^0 - L_p^0 - N_r^0$$
$$C = L_p^0 N_r^0 - N_p^0 L_r^0 + Y_v^o N_r^0 + Y_v^0 L_p^0 - L_v^0 Y_p^0 - N_v^0 Y_r^0 + N_v V_{x0} - L_v^0 V_{z0}$$
$$D = -L_v^0 g\cos\Theta_0 - Y_v^0 L_p^0 N_r^0 + Y_v^0 L_r^0 N_p^0 + Y_p^0 L_v^0 N_r^0 - Y_r^0 L_v^0 N_p^0 - Y_p^0 L_r^0 N_v^0$$
$$+ Y_r^0 L_p^0 N_v^0 + (L_v^0 N_p^0 - L_p^0 N_v^0) V_{x0} + (L_v^0 N_r^0 - L_r^0 N_v^0) V_{z0}$$
$$E = (L_v^0 N_r^0 - L_r^0 N_v^0) g\cos\Theta_0$$
$$F = 0$$

As in the hover, the characteristic equation has one root (λ=0) relative to a neutral condition. Then, generally it presents other two real roots and two complex roots, as follows:

— a rolling damped motion (the *roll mode*) and a spiral motion (the *spiral mode*) correspond to the two real roots;

— a lateral-directional oscillation, LDO (therefore, an oscillation in roll and in yaw, with a pitching), called *'dutch-roll'*, corresponds to the pair of complex roots, similar to the motion defined for the fixed wing aircraft.

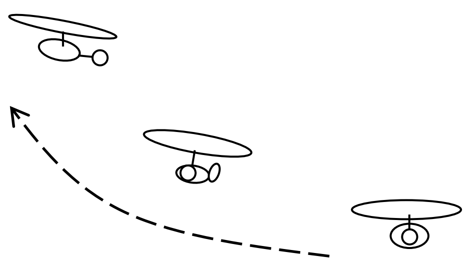

Figure 7.10 Dutch-roll oscillation

Example of lateral-directional root locus plot as a function of forward flight speed

Figure 7.11 illustrates the typical influence of the forward speed, from the hovering flight to high speed flight, on the lateral-directional eigenvalues for a medium helicopter.

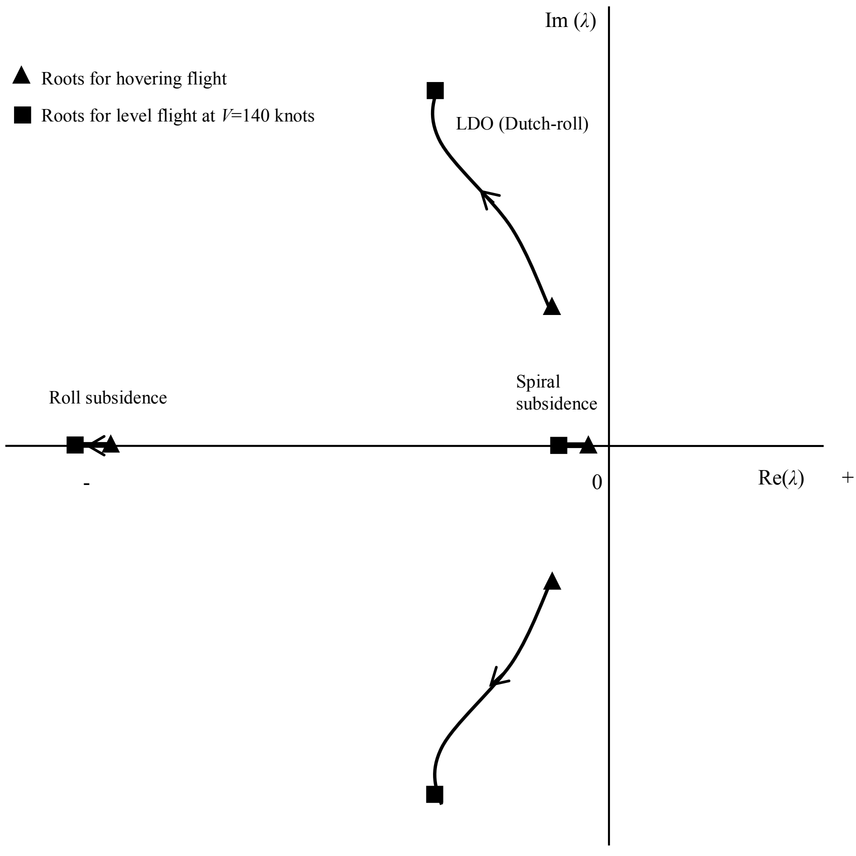

Figure 7.11 Example of lateral-directional root locus as a function of forward speed

7.8. Helicopter control

7.8.1. *Stability, control and flying qualities*

The study of helicopter control is based on the analysis of the whole aircraft response following the application of one or more control inputs by the 'human' pilot or by an automatic system (auto-pilot).

Through the flight controls, the pilot shall perform a flight manoeuvre or shall compensate adequately an atmospheric disturbance (as it can be a gust). Then, it is necessary to verify the helicopter response in the entire spectrum of manoeuvre and for each aircraft configuration.

Moreover, it shall be noted that stability and manoeuvrability (capability of rapid response to a pilot action) are substantially opposite characteristics, because an aircraft with high stability 'suffers' from low manoeuvrability. From this standpoint, required stability and manoeuvrability will vary with the helicopter type and with the flight mission requirements.

Generally, the reference specifications for stability and control requirements of V/STOL aircraft define various classes versus weights and manoeuvrability (light, heavy, low/medium and high manoeuvrability).

Typically, the specifications define also the flying qualities in terms of defined levels, related to the capability to complete the flight mission.
For example, typical levels are described as follows:

Level 1	flying qualities are adequate
Level 2	flying qualities are adequate, but increment in pilot work load is required
Level 3	flying qualities allow the helicopter to fly safely, but intense pilot work load is required

7.8.2. *Longitudinal control in hovering flight; one degree of freedom approach*

The application of a control by the pilot requires (in general, both in hovering and in forward flight) an additional action of compensation through other controls in order to perform correctly a manoeuvre (in some cases, also an appropriate mix of controls to minimize the secondary effects is present). For example, a change in collective pitch in hover generates, of course, a change in rotor thrust (the primary effect), but causes also a change in rotor torque that shall be counteracted acting through the pedals in order to maintain the flight direction.

After this introduction (to be always present for an entire and advanced analysis), we will obtain very useful information also by simplified models. In particular, in the pages that follow we will analyze some cases of helicopter response by an approach with only one degree of freedom.

Therefore, we can use Equations (7.10) with the application of the only one forcing B_1 and with the assumption of one degree of freedom about the pitching axis. Then, in order to analyze the helicopter attitude, the third equation of (7.10) can be written as:

$$-\dot{q}I_y + M_q q = -M_{B_1} B_1$$

Dividing by I_y, using the normalized notation adopted for the relationships (7.11), and going into Laplace domain (Appendix C), we obtain:

$$\frac{q}{B_1} = \frac{M^0_{B_1}}{(s - M^0_q)} \tag{7.16}$$

The expression (7.16) is *the transfer function* of the pitch rate q due to the longitudinal cyclic pitch B_1.

Fixing the forcing changes and performing the inverse transformation, finally we have the relationship between q and B_1 in time domain.

In order to complete the treatment, note that

$$M_{B_1} = \frac{\partial M}{\partial B_1} = \frac{dM_{MR}}{da_{1S}} \frac{\partial a_{1S}}{\partial B_1} - \left(\frac{\partial X}{\partial B_1} \right)_{MR} \cdot h_z$$

Now, let us consider the case where a *vertical gust* of magnitude w_{gust} is developed; then, consider the second equation of the group of Equations

(7.10), and by introducing the disturbance w_{gust} and taking into account only the one degree of freedom to the vertical movement, we obtain:

$$-\frac{W_G}{g}(\dot{w})+Z_w\left(w+w_{gust}\right)=0$$

and finally

$$\dot{w}-Z_w^0 w=Z_w^0 w_{gust}$$

As in the previous case, if we know the changes in disturbance, then we can calculate the approximate response of the system by means of the described methods.

Note that it is:

$$Z_W=\frac{\partial Z}{\partial w}=-\frac{\partial T_{Tpp}}{\partial w}$$

7.8.3. *Lateral-directional control in hovering flight; one degree of freedom approach*

If we consider the forcing θ_{tr} and only the degree of freedom about the yawing axis, the third equation of set (7.13) can be written as:

$$\dot{r}I_z=N_r r+N_{\theta_{tr}}\theta_{tr}$$

obtaining

$$\frac{r}{\theta_{tr}}=\frac{N_{\theta_{tr}}^0}{(s-N_r^0)} \tag{7.16}$$

By using the previous assumption, the relation (7.16) describes the response following the forcing (collective control input for the tail rotor).

In particular, if we assume that the forcing changes according to the following rule:

$$\theta_{tr}(t) = \begin{cases} 0 & t \leq 0 \\ \theta_{st} u_{step}(t) & t > 0 \end{cases}$$ (step function, *step change input*)

and then

$$\theta_{tr}(s) = \frac{\theta_{st}}{s}$$

finally, in the time domain we obtain:

$$\frac{r(t)}{\theta_{st}} = -\frac{N_{\theta_{tr}}}{N_r}\left(1 - e^{\frac{N_r}{I_z}t}\right)$$ (for $t > 0$).