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On Minimal Non-Soluble Groups, the Normalizer Condition and McLain Groups *

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Abstract

We prove that a minimal non-soluble ($MN\mathfrak{S}$ in short) Fitting p-group G has a proper subgroup K such that for every proper subgroup R of G containing K, we have $N_G(R) > R$. In other words, G satisfies the normalizer condition modulo K. We also give a positive answer in McLain groups to a question aroused from the works on $MN\mathfrak{S}$ Fitting p-groups.

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1 Introduction

Let G be a group and $\mathfrak{X}$ a class of groups. If $G \notin \mathfrak{X}$ but for every proper subgroup K of G we have $K \in \mathfrak{X}$, then G is called a minimal non $\mathfrak{X}$-group and is usually denoted by $MN\mathfrak{X}$. In the present paper we consider $MN\mathfrak{S}$ Fitting p-groups, where $\mathfrak{S}$ denotes the class of all soluble groups. Locally finite $MN\mathfrak{S}$-groups are considered mainly in [1]–[6] and it is not known yet if such groups (in particular, Fitting p-groups) exist and it is also not yet known whether they must satisfy the normalizer condition. In the present note we obtain the following partial answer:

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Theorem 1 *Let* G *be an* $MN\mathfrak{S}$ *Fitting* p*-group, then* G *has a proper subgroup* K *such that for every proper subgroup* R *of* G *containing* K*, we have* $N_G(R) > R$*. In other words,* G *satisfies the normalizer condition modulo* K*.*

In [6], $MN\mathfrak{S}$ Fitting p-groups satisfying the normalizer condition are considered and it is shown that such groups with some additional conditions do not exist.

Let G be a group, K be a subgroup of G and $x \in G$. We define *the center modulo* K as

$$\begin{aligned} Z(G//K) :=& \{g \in G \mid [g,x] \leqslant K \text{ for every } x \in G\} \\ =& \{g \in G \mid [g,G] \leqslant K\} \end{aligned}$$

which is a subgroup of G, and *the centralizer of* x *modulo* K as

$$X_G(xK) := \langle g \in G \mid [g,x] \in K \rangle$$

which will be useful in the sequel. If we take $Z(G//K) = Z$, then Z is normal in G and $Z(G/Z) = 1$, whenever G is perfect. But if $K^G = G$, then K is not contained in Z. This makes some difficulties (namely in the proof of Lemma 4), so we should work with subgroups modulo K.

In the present note we also answer the following question positively:

> Does there exist a locally nilpotent perfect group G having a proper subgroup V which does not normalize any non-trivial finitely generated subgroup of G?

The question arises from [1] with the work on minimal non-soluble Fitting-p-groups. In [1], it is seen that such groups have a proper subgroup which satisfies the property in question. We wondered that such groups really exist. In the present work we shall show that the McLain group $M := M(\mathbb{Q}, F)$ for a field F satisfies the property in question (see [11, p.361]), as the following theorem states:

Theorem 2 *The group* M *has a proper subgroup* V *which does not normalize any non-trivial finitely generated subgroup of* M*. In particular,* $C_M(V) = \{1\}$*.*

2 Proof of Theorem 1

In [1] and [5], some versions of the following lemma have been proved. However, next statement does not include any imposition to homomorphic images, so that it is a direct result.

Lemma 3 *Let* G *be an* $\mathrm{MN}\mathfrak{S}$ *Fitting* p*-group for some prime* p*. Then for every finite subgroup* U *and every proper subgroup* L*, we have*

$$\bigcap_{y\in G\setminus L} \langle U, y\rangle = U.$$

Proof — Assume that the assertion is false and so

$$\bigcap_{y\in G\setminus L} \langle U, y\rangle \neq U$$

for some finite subgroup U and a proper subgroup of L of G. Take

$$a \in \Big(\bigcap_{y\in G\setminus L} \langle U, y\rangle\Big) \setminus U.$$

Now, G has an ascending sequence of finite subgroups F_i, for $i \geqslant 1$, such that

$$G = \bigcup_{i\geqslant 1} F_i^G.$$

For all $i \geqslant 1$, $N_i := F_i^G$ is nilpotent of finite exponent. Since L is soluble and G is perfect, we have that there is a positive integer r such that $N_r/(N_r \cap L)N_r'$ is infinite. We may also assume that $\langle a, U\rangle \leqslant N_r$. Put

$$S_r := (N_r \cap L)N_r' \quad \text{and} \quad K/S_r := \mathrm{Frat}(N_r/(N_r \cap L)N_r').$$

Now N_r/K is infinite elementary abelian. By [8, Satz 6], N_r has a subgroup V such that $U \leqslant V$, VK/K is infinite and $a \notin V$. Hence there is an element $z \in V \setminus K$ such that $a \notin \langle U, z\rangle$. But since $N_r \cap L \leqslant K$, we have that $z \notin L$, and this is a contradiction. □

Define $\delta_0(x) = x$ and

$$\delta_n(x_1, x_2 \ldots, x_{2^n}) = [\delta_{n-1}(x_1, x_2 \ldots, x_{2^{n-1}}), \delta_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})]$$

for $n \geqslant 1$. Then a group G is soluble of derived length d if and only if $\delta_d(g_1, g_2 \dots, g_{2^d}) = 1$ for all $g_1, g_2 \dots, g_{2^d} \in G$.

Inspired by [9, Lemma 4] we prove the following critical lemma.

Lemma 4 *Let* G *be an* $MN\mathfrak{G}$ *Fitting* p*-group for some prime* p*. Then* G *has a proper subgroup* K *such that* $Z(G//R) \nleqslant R$ *whenever* $K \leqslant R < G$*. In other words,* $X_G(gR) = G$ *for some* $g \in G \setminus R$*.*

Proof — Assume that the assertion is false. We shall prove that for every proper subgroup K and for every finite subgroup U of G, for $a \in G \setminus U$ and for every outer commutator word $\psi(x_1, \dots, x_n)$, $n \geqslant 1$, there exist elements $y_1, \dots, y_n \in G$ such that $\psi(y_1, \dots, y_n) \notin K$ and $a \notin \langle U, y_1, \dots, y_n \rangle$. By Lemma 3, we have

$$\bigcap_{y \in G \setminus K} \langle U, y \rangle = U$$

and thus there is $y_1 \in G \setminus K$ such that $a \notin \langle U, y_1 \rangle$ and $\psi(y_1) = y_1 \notin K$. So the assertion is true for $n = 1$. By assumption, G has a proper subgroup R such that $Z(G//R) \leqslant R$. Let

$$\psi(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = [\varphi(x_1, \dots, x_m), \chi(x_{m+1}, \dots, x_n)].$$

By induction hypothesis there are $y_1, \dots, y_m \in G$ such that

$$a \notin \langle U, y_1, \dots, y_m \rangle \quad \text{and} \quad \varphi(y_1, \dots, y_m) \notin R.$$

Now $X_G(\varphi(y_1, \dots, y_m)R) \neq G$ by assumption. By induction hypothesis G has elements $y_{m+1}, \dots, y_n$ such that

$$a \notin \langle U, y_1, \dots, y_m, y_{m+1}, \dots, y_n \rangle$$

and

$$\chi(y_{m+1}, \dots, y_n) \notin X_G(\varphi(y_1, \dots, y_m)R).$$

Hence

$$\begin{aligned} &\psi(y_1, \dots, y_m, y_{m+1}, \dots, y_n) \\ &= [\varphi(y_1, \dots, y_m), \chi(y_{m+1}, \dots, y_n)] \notin R \geqslant K. \end{aligned}$$

So the induction is complete.

By the above argument, we can find elements

$$y_{1,1}, y_{1,2}; \dots ; y_{i,1}, y_{i,2}, \dots, y_{i,2^i}; \dots$$

in G such that

$$a \notin X := \langle y_{i,1}, y_{i,2}, \dots, y_{i,2^i} \mid i \geqslant 1 \rangle \quad \text{and} \quad \delta_i(y_{i,1}, y_{i,2}, \dots, y_{i,2^i}) \neq 1$$

for every $i \geqslant 1$. But then X is a non-soluble proper subgroup of G, which is a contradiction. □

Proof of Theorem 1 — Let K be the subgroup defined in the statement of Lemma 4 and let R be a proper subgroup of G containing K. Then by Lemma 4, there is $g \in G \setminus R$ such that $X_G(gR) = G$. Then $[g, G] \leqslant R$ and in particular $[g, R] \leqslant R$. This means that g belongs to $N_G(R) \setminus R$. Hence the theorem is proved. □

3 Proof of Theorem 2

Throughout this section, let $M := M(\mathbb{Q}, F)$ for a field F. Then M is a characteristically simple, locally nilpotent group (see [11, 12.1.9]). Hence $Z(M) = 1$ and M is perfect. Furthermore, if F has characteristic 0, then M is torsion-free and if F has characteristic p for some prime p, then M is a p-group. Also M is not finitely generated and has no proper subgroup of finite index.

Let us consider the following definitions, which are given in [7] and will be used in the sequel.

Let $g = 1 + \sum_{\lambda,\mu} c_{\lambda,\mu} e_{\lambda,\mu} \in M$. Define

$$[g] = \{(\lambda, \mu) : \ c_{\lambda,\mu} \neq 0\},$$

the *support* of g,

$$[g]_1 = \{\lambda \in \mathbb{Q} : \text{ there exists } \mu \in \mathbb{Q} \text{ such that } (\lambda, \mu) \in [g]\},$$

the 1-*support* of g, and

$$[g]_2 = \{\mu \in \mathbb{Q} : \text{ there exists } \lambda \in \mathbb{Q} \text{ such that } (\lambda, \mu) \in [g]\},$$

the 2-*support* of g.

Let R be a subgroup of M. We define

$$S(R) := \Big(\bigcup_{g\in R}[g]_1\Big) \cup \Big(\bigcup_{g\in R}[g]_2\Big).$$

Before embarking on the proof of Theorem 2, but first we give some properties of M.

Lemma 5 *For every finitely generated subgroup F and every proper subgroup K of M, we have*

$$\langle F, K\rangle \neq M.$$

Proof — Assume that $M = \langle F, K\rangle$ for a non-trivial finitely generated subgroup F and a proper subgroup K of M. Now, we may assume that $M = \langle K, x\rangle$ for some $x \in G \setminus K$. But this implies that M has a maximal subgroup, a contradiction. □

Lemma 6 *For every finitely generated subgroup U and every proper subgroup L of M, we have*

$$\bigcap_{y\in M\setminus L} \langle U, y\rangle = U.$$

Proof — Assume that the result is false. Then there is a finitely generated subgroup U and a proper subgroup L of G such that

$$\bigcap_{y\in M\setminus L} \langle U, y\rangle \neq U$$

and hence there is an element

$$a \in \Big(\bigcap_{y\in M\setminus L} \langle U, y\rangle\Big) \setminus U.$$

Let $a = 1 + c_1 e_{\sigma_1,\tau_1} + \cdots + c_s e_{\sigma_s,\tau_s}$. Clearly, L does not contain all generators of M of the form $1 + de_{\alpha,\beta}$. Since $a \notin U$ and $a \in \langle U, y\rangle$ for every $y \in M \setminus L$, such a generator which is not contained in L must be of the form $1 + ge_{\sigma_i,\delta}$ or $1 + he_{\gamma,\tau_i}$ for some $1 \leqslant i \leqslant s$. Since these generators must supplement the elements of U to generate a, we have $\delta, \gamma \in S(U)$, i.e. $M \setminus L$ contains only finitely many generators of M of the form $1 + de_{\alpha,\beta}$. This yields $M = \langle F, L\rangle$ for some finite set F. But this contradicts Lemma 5. □

Theorem 7 *If $\{S_i \mid i \geqslant 1\}$ is a set of proper subgroups of M, then M has a proper subgroup V such that $V \nleq S_i$ for all $i \geqslant 1$.*